College of Engineering and Computer Science
Mechanical Engineering Department

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Mechanical Engineering 501A
Seminar in Engineering Analysis

Fall 2017 Number 15966 Instructor: Larry Caretto

September 6 Homework Solutions

1. A vector originating at the point (-3, -8, 1) has components 3, 8, -1. Find the terminating point and the length of the vector.

The terminating point is (-3+ 3 = 0, -8 + 8 = 0, 1 +(-1) = 0).

The length of the vector is the square root of the sum of the squares of its components =

$$\sqrt{3^2 + 8^2 + (-1)^2} = \sqrt{9 + 64 + 1} = \sqrt{74}$$

2. A vector originating at the point (1, ½, -4) has components ¼, ½, -½. Find the terminating point and the length of the vector.

The terminating point is (1+ ¼ = 5/4, ½ + ½ = 1, -4 + ½ = -4½).

The length of the vector is the square root of the sum of the squares of its components =

$$\sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1+4+4}{16}} = \frac{3}{4}$$

3. Given a **= [3,-2,1] and** b **= [0, 3, 0], find** |a + b**| and** |a**|** + |b**|.**

|a + **b|** = | [3,-2,1] + [0, 3, 0] | = | [3+0, -2+3, 1+0] | = | [3, 1, 1] | = $\sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}$

|a| + **|b|** = | [3,-2,1] |+ | [0, 3, 0] | = $\sqrt{3^2 + (-2)^2 + 1^2} + \sqrt{0^2 + 3^2 + 0^2} = \sqrt{14} + 3$

This is a specific example of the general rule that **|a** + **b|** ≤ **|a|** + **|b|**.

4. Given a **= [3,-2,1] and** b **= [0, 3, 0], find the expressions below.**

From problem 3, **|a|** = $\sqrt{14}$ and **|b|** = 3.

a/|**a|** = [3, 2, 1]/ $\sqrt{14}$ =[3/ $\sqrt{14}$, –2/ $\sqrt{14}$, 1/ $\sqrt{14}$]

b/|**b|** =[0, 3, 0] / 3 = [0, 1, 0]

5. **Given** a **= [1, 1, 0],** b **= [3, 2, 1], and** c **= [1, 0, 2], find the angles between** b **–** a **and** c – a.

Here we use the result that for any two vectors, **x** and **y**, the dot product of the two vectors, **x•y**, equals **|x||y|**cos(θ), where θ is the angle between the two vectors.

b – **a** = [3, 2, 1] – [1, 1. 0] = [2, 1, 1] so **|b** – **a|** = $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

c – **a** = [1, 0, 2] – [1, 1. 0] = [0, -1, 2] so **|b** – **a|** = $\sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5}$

(**b** – **a**)•(**c** – **a**) = (2)(0) + (1)(-1) +(1)(2) = 1

$$\theta = \cos^{-1}\left(\frac{(\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{c}-\mathbf{a}||\mathbf{b}-\mathbf{a}|}\right) = \cos^{-1}\left(\frac{1}{(\sqrt{6})(\sqrt{5})}\right) = \cos^{-1}\left(\frac{1}{(\sqrt{30})}\right)$$ =1.387…= 79.48…degrees

6. Find the angle between straight lines x – y = 1 and x – 2y = -1

Here we use the result that for any two vectors, **x** and **y**, the dot product of the two vectors, **x•y**, equals **|x||y|**cos(θ), where θ is the angle between the two vectors, but we have to represent the two lines as vector components.

We want the vector components for the line, but we are not sure how long the line is. However, when we are only interested in the angle between the two lines, we can use *relative* components for the vectors that represent the lines, so long as the two sets of components are consistent with each other. We can do this if we arbitrarily take the x component to be 1 and the y component to be the slope of the line.

For the first line, written as y = x – 1, we see that the slope is one, so that if the x component of a vector along this line has a value of α, the y component would also have a component of α, because the slope is one. Thus, we could write this vector as [α α]. For the second line, written as y = (x + 1)/2, we see that the slope is ½. Thus, if the x component of a vector along this line is β, the y component of this vector is β/2. Thus our second vector, lying along the second line, can be written as [β β/2]

The length of the first vector is $\sqrt{\alpha^2+\alpha^2} = \sqrt{2}\alpha.$ For the second vector, the length is $\sqrt{\beta^2+\left(\frac{\beta}{2}\right)^2} = \frac{\sqrt{5}}{2}\beta.$ The dot product of the two vectors is [α α]∙[β β/2] = (α)(β) + (α)(β/2) = 3αβ/2. We can then find the angle between the two lines from the relation for the dot product of two vectors.

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{\frac{3\alpha\beta}{2}}{(\sqrt{2}\alpha)\left(\frac{\sqrt{5}\beta}{2}\right)}\right) = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$$ =0.32175…= 18.43…degrees

7. Find the component of a **= [4, 0, -3] in the direction of** b **= [1, 1, 1].**

Equation (11) on page 412 gives the general equation for finding the (scalar) component, p, of a vector, **a**, in the direction of a vector **b**, as p = **a∙b**/|**b**|. For this problem, **a∙b** = (4)(1) + (0)(1) + (-3)(1) = 1, and |**b**| = $\sqrt{1^2+1^2+1^2} = \sqrt{3}$. Thus, p = **a∙b**/|**b**|. = $1/\sqrt{3}$.

8. Find the matrix sums 5D **– 3**C **and 5**D^T **– 3**C^T **for the matrices shown below.**

$$5D-3C = 5\begin{bmatrix}4 & 0 & -4\\ -3 & 4 & 9\end{bmatrix} - 3\begin{bmatrix}6 & 0 & 3\\ 1 & 0 & -5\end{bmatrix} = \begin{bmatrix}20-18 & 0-0 & -20-9\\ -15-3 & 20-0 & 45+15\end{bmatrix} = \begin{bmatrix}2 & 0 & -29\\ -18 & 20 & 60\end{bmatrix}$$

$$5D^T-3C^T = 5\begin{bmatrix}4 & -3\\ 0 & 4\\ -4 & 9\end{bmatrix} - 3\begin{bmatrix}6 & 1\\ 0 & 0\\ 3 & -5\end{bmatrix} = \begin{bmatrix}20-18 & -15-3\\ 9\ \ 0-0 & 29-0\\ -20-9 & 45+15\end{bmatrix} = \begin{bmatrix}2 & -18\\ 0 & 20\\ -29 & 60\end{bmatrix}$$

This is a specific example of the general result that $(\alpha\mathbf{A} \pm \beta\mathbf{B})^T = \alpha\mathbf{A}^T \pm \beta\mathbf{B}^T$.

9. Find the vector sums 3(c – 4d) and 3c – 12d for the vectors shown below.

$$3(\mathbf{c}-4\mathbf{d}) = 3\left(\begin{bmatrix}9\\5\\7\end{bmatrix} - 4\begin{bmatrix}2\\-2\\6\end{bmatrix}\right) = 3\left(\begin{bmatrix}9\\5\\7\end{bmatrix} - \begin{bmatrix}8\\-8\\24\end{bmatrix}\right) = 3\begin{bmatrix}1\\13\\-17\end{bmatrix} = \begin{bmatrix}3\\39\\-51\end{bmatrix}$$

$$3\mathbf{c}-12\mathbf{d} = 3\begin{bmatrix}9\\5\\7\end{bmatrix} - 12\begin{bmatrix}2\\-2\\6\end{bmatrix} = \begin{bmatrix}27\\15\\21\end{bmatrix} - \begin{bmatrix}24\\-24\\72\end{bmatrix} = \begin{bmatrix}3\\39\\-51\end{bmatrix}$$

This is an example of the general rule that $\alpha(\mathbf{c} - \beta\mathbf{d}) = \alpha\mathbf{c} - \alpha\beta\mathbf{d}$, where α and β are any two arbitrary scalars.

10. Find the matrix products, if they exist, Ca, Cd, and dC, for the matrices shown below.

$$\mathbf{Ca} = \begin{bmatrix}4 & 6 & 2\\6 & 0 & 3\\2 & 3 & -1\end{bmatrix}\begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}(4)(1)+(6)(4)+(2)(3)\\(6)(1)+(0)(4)+(3)(3)\\(2)(1)+(3)(4)+(-1)(3)\end{bmatrix} = \begin{bmatrix}34\\15\\11\end{bmatrix}$$

The matrix product **Cd** is not possible. This calls for the premultiplication of a (1 x 3) matrix by a (3 x 3) matrix. The number of columns in the premultiplying matrix (3) does not match the number of rows in the postmultiplication matrix (1).

$$\mathbf{dC} = \begin{bmatrix}4 & 3 & 0\end{bmatrix}\begin{bmatrix}4 & 6 & 2\\6 & 0 & 3\\2 & 3 & -1\end{bmatrix} = \begin{bmatrix}4(4)+3(6)+0(2) & 4(6)+3(0)+0(3) & 4(2)+3(3)+0(-1)\end{bmatrix}$$

$$\mathbf{dC} = \begin{bmatrix}4(4)+3(6)+0(2) & 4(6)+3(0)+0(3) & 4(2)+3(3)+0(-1)\end{bmatrix} = \begin{bmatrix}34 & 24 & 17\end{bmatrix}$$

11. Evaluate the determinant shown below.

Use the expansion formula in equation (9b) on page 343 to expand the determinant in terms of the cofactors of the first column of the determinant

$$\begin{vmatrix}3 & 2 & 0 & 0\\6 & 8 & 0 & 0\\0 & 0 & 4 & 7\\0 & 0 & 2 & 5\end{vmatrix} = 3(-1)^{1+1}\begin{vmatrix}8 & 0 & 0\\0 & 4 & 7\\0 & 2 & 5\end{vmatrix} + 6(-1)^{1+2}\begin{vmatrix}2 & 0 & 0\\0 & 4 & 7\\0 & 2 & 5\end{vmatrix} + 0(-1)^{1+3}\begin{vmatrix}2 & 0 & 0\\8 & 0 & 0\\0 & 2 & 5\end{vmatrix} + 0(-1)^{1+4}\begin{vmatrix}2 & 0 & 0\\8 & 0 & 0\\0 & 4 & 7\end{vmatrix}$$

$$= 3\left[(8)(4)(5)+(0)(2)(0)+(0)(0)(7)-(0)(4)(0)-(0)(0)(5)-(8)(2)(7)\right]$$
$$-6\left[(2)(4)(5)+(0)(2)(0)+(0)(0)(7)-(0)(4)(0)-(0)(0)(5)-(2)(7)(2)\right]+0[\cdots]-0[\cdots]$$
$$= 3[160+0+0-0-0-112]-6[40+0+0-0-0-28] = 144-72 = 72$$

12. Show that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$.

Start with the basic definition that a matrix times its inverse, $\mathbf{AA}^{-1} = \mathbf{I}$, and take the transpose of both sides. For the unit matrix, $\mathbf{I}$, $\mathbf{I}^T = \mathbf{I}$. For the left side of the equation, we apply the rule for the transpose of a matrix product, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

$$\mathbf{AA}^{-1} = \mathbf{I} \quad => \quad (\mathbf{AA}^{-1})^T = \mathbf{I}^T = \mathbf{I} \quad => \quad (\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}$$

The next step is to postmultiply both sides of the equation by $(\mathbf{A}^T)^{-1}$. After doing this we recognize that the produce of a matrix and its inverse is the unit matrix and the product of any matrix, $\mathbf{B}$, and the unit matrix is simply the original matrix, $\mathbf{B}$. This gives the following steps.

$$(\mathbf{A}^{-1})^T\mathbf{A}^T(\mathbf{A}^T)^{-1} = \mathbf{I}(\mathbf{A}^T)^{-1} \quad => \quad (\mathbf{A}^{-1})^T\mathbf{I} = (\mathbf{A}^T)^{-1} \quad => \quad (\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$$

This completes the proof.