Power Series Solutions and Frobenius Method

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Outline
- Review last week
- Power series solutions
  - General approach
  - Application
- Frobenius method
  - Basic process
  - Application to Bessel’s Equation
- Review for midterm on Wednesday

Review last week
- Systems of differential equations derived from physical problems involving different, interacting parts
- Showed how to convert two second order equations into one fourth order equation
  - Solved fourth order equation for one variable then used algebra for second
  - Fit initial conditions on both variables using both solutions

Review Last Week II
- Solved system of linear first order differential equations
  \[ \frac{dy_i}{dt} + \sum_{j=1}^{n} a_{ij} y_j = r_i(t) \quad i = 1, \ldots, n \]
  \[ \frac{dy}{dt} + A y = r \]
- Solved in terms of eigenvalues, \( \lambda \), and eigenvector matrix, \( X \), for \( A \)

Review Last Week III
- Define new variable, \( s_i = \mathbf{X}^{-1} y_i \) \((y = \mathbf{X} s)\)
- Transform original equation as follows
  \[ \mathbf{X}^{-1} \frac{d\mathbf{s}}{dt} + \mathbf{X}^{-1} \mathbf{A} \mathbf{s} = \mathbf{X}^{-1} \mathbf{r} \quad \Rightarrow \quad \frac{d\mathbf{s}}{dt} + \mathbf{\lambda} \mathbf{s} = \mathbf{X}^{-1} \mathbf{r} = \mathbf{p} \]
- Transformed equation is scalar equation whose solution is known
  \[ s_i = e^{-\lambda_i t} \left[ e^{\lambda_i t} p_i dt + C_i \right] = C_i e^{-\lambda_i t} + q_i \]

Review Last Week IV
- Definitions to convert \( s_i = C_i e^{\lambda_i t} + q_i \) into matrix equation \( s = \mathbf{E}(t) \mathbf{C} + \mathbf{q} \) \((\mathbf{E}(0) = 1)\)
  \[ \mathbf{E}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{-\lambda_1 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{-\lambda_1 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{-\lambda_1 t} \end{bmatrix} \]
  \[ \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_s \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \]
Review Last Week V

• Convert matrix equation for $s$ into matrix equation for $y$ using $y = Xs$
• Apply initial condition that $y = y_0$ at $t = 0$, (where $E = I$): $C = X^{-1}y_0 - q_0$
• Result: $y = XE\left[X^{-1}y_0 - q_0\right] + Xq$
• Homogenous ($q = 0$): $y = XEX^{-1}y_0$

Review ODE Solution Basis

• A homogenous $n$th order ODE has a basis of $n$ linearly independent solutions
• $d^2y/dx^2 - k^2y = 0$ has the following possible solutions: $e^{kx}$, $e^{-kx}$, sinh$(kx)$, and cosh$(kx)$ but only two of these are linearly independent
• Have to find complete basis to be able to represent all possible initial or boundary conditions

Power Series Solutions

• Look at following differential equation and proposed power series solution
• Requires $p(x)$, $q(x)$ and $r(x)$ that can be expanded in power series about $x = x_0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Getting the Solutions

• Differentiate power series solution and substitute it into differential equation

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} + p(x) \frac{dy}{dx} + q(x)y = r(x)$$

$$\frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x)$$

$$\sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} + q(x) \sum_{n=0}^{\infty} a_n (x - x_0)^n = r(x)$$

• Look at simple example with $p(x) = r(x) = 0$ and $q(x) = k^2$

Getting the Solutions II

$$\sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2} = 0 + 0 + \sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$$

First sum now has same $(x - x_0)^n$ factor and $0$ to $\infty$ limits as second sum

$$\sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2} + k^2 \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} [n + (n+1)]a_{n+1}(x - x_0)^n$$
Making Coefficients Vanish

\[ \sum_{n=0}^{\infty} \left[ (n+2)(n+1)a_{n+2} + k^2a_n \right] x^n = 0 \]

- Each coefficient of \((x - x_0)^n\) vanishes if 
  \((n + 2)(n + 1)a_{n+2} + k^2a_n = 0\)

- Gives recursion equation for \(a_n\)
  \[ a_{n+2} = -\frac{k^2a_n}{(n+2)(n+1)} \]

Want \(a_n\) as function of \(n\)

Check \(a_n\) Equation

- Write general \(a_n\) equation for \(a_{n+2}\) then check ratio \(a_{n+2}/a_n\)
  \[ a_{n+2}/a_n = \frac{(n+2)!}{(-1)^{n+2}k^n a_0} \]

- Proposed \(a_n\) equation gives same result for \(a_{n+2}/a_n\) derived from power series

Repeat Process for odd \(a_n\)

- All odd \(a_n\) proportional to \(a_1\)
- Original solution now has two series
  - Solutions are expected power series for sine and cosine
  - \(a_0\) and \(a_1\) chosen to fit initial conditions

\[ y(x) = a_0 \left[ 1 - \frac{k(x - x_0)^2}{2!} + \frac{k(x - x_0)^4}{4!} - \cdots \right] \]
\[ + a_1 \left[ \frac{k(x - x_0)}{3!} - \frac{k(x - x_0)^3}{5!} - \cdots \right] \]

Summary for

\[ \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x) \]

- Write the solution for \(y(x)\) as a power series in unknown coefficients
  \[ y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

- Differentiate the power series to get the derivatives required in the differential equation

\[ \frac{dy}{dx} = \sum_{n=1}^{\infty} na_n (x - x_0)^{n-1} \]
\[ \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)a_n (x - x_0)^{n-2} \]

- Get series for \(p(x)\), \(q(x)\), \(r(x)\) if required
- Substitute into differential equation

Summary Continues

- Rewrite the resulting equation to group terms with common powers of \(x - x_0\).
- Set the coefficients of each power of \(x - x_0\) equal to zero giving an equation relating neighboring values of \(a_n\)
- Relate coefficients with higher subscripts to those with lower subscripts.
- Initial unknown coefficients, e.g., \(a_0\), \(a_1\), etc., are found from initial conditions
Summary

Concludes

• Examine equations relating neighboring coefficients and try to obtain a general equation for each an in terms of the unknown coefficients a₀, a₁, etc.
• Substitute the general expression for aₙ into the original power series for y(x)
• This is the final power series solution

Frobenius Method

• Applied to differential equation below around x₀ = 0 where b(x)/x and c(x)/x² make usual power series method inapplicable

\[ \frac{d^2y(x)}{dx^2} + \frac{b(x)}{x} \frac{dy(x)}{dx} + \frac{c(x)}{x^2} y = 0 \]

• Solution similar to previous power series (with x₀ = 0) except for xʳ factor

\[ y(x) = x^r \sum_{n=0}^{\infty} a_n x^{n+r} \]

Frobenius Method II

• Differentiate proposed solution two times
• Get power series for b(x) and c(x)
• Substitute into original equation

\[ \frac{d^2y(x)}{dx^2} + \frac{b(x)}{x} \frac{dy(x)}{dx} + \frac{c(x)}{x^2} y = 0 \]

Frobenius Method III

• Manipulate to get single summation with common power of x in each term

\[ \frac{d^2y(x)}{dx^2} + \frac{b(x)}{x} \frac{dy(x)}{dx} + \frac{c(x)}{x^2} y = 0 \]

\[ \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + \frac{\sum b_n x^r}{x} + \frac{\sum c_n x^r}{x^2} \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \]

• Manipulate to get single summation with common power of x in each term

Frobenius Method IV

• Multiply result by x² and combine x and x² factors with xⁿ+r terms in sums

\[ \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \left( \sum_{n=0}^{\infty} b_n x^r \right) \cdot B \]

\[ \left( \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} \right) + \left( \sum_{n=0}^{\infty} c_n x^r \right) \left( \sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \]

• Expand series and multiply term by term to get first few terms in the series for the case where b(x) = b₀ and c(x) = c₀

Frobenius Method V

A \[ r(r-1)a_n x^r + (1+r)ra_n x^{r+1} + (2+r)(1+r)a_n x^{r+2} + \cdots \]

+ \[ b₀ x^r + b₁(1+r)a_n x^{r+1} + b₂(2+r)(1+r)a_n x^{r+2} + \cdots \]

B \[ b₂x_r + b₃(1+r)a_n x^{r+1} + b₄(2+r)(1+r)a_n x^{r+2} + \cdots \]

C \[ c₃a_n x^r + c₄a_n x^{r+1} + c₅a_n x^{r+2} + \cdots = 0 \]

• Coefficients of xʳ term must vanish
  – r(r-1)a₀ + b₁r + c₂a₀ = 0
  – Do not want a₀ = 0
  – This requires r(r-1) + b₀r + c₀ = 0
**Frobenius Method V**

- Equation \( r(r - 1) + D_0 r + C_0 = 0 \) is known as indicial equation.
- It is a quadratic equation giving two solutions for (index) \( r \):
  \[
  r = \frac{1}{2} \left( \frac{-b \pm \sqrt{b^2 - 4c}}{2} \right)
  \]
- Choose higher value of \( r \) for first solution.
- Second ODE solution depends on \( r \) values:
  - Double root, roots differing by an integer,
  - Roots differing by a non-integer.

**Frobenius Method VI**

- First and second solutions \( y_1(x) \) and \( y_2(x) \):
  \[
  y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n
  \]
  \[
  y_2(x) = x^{r_1} \sum_{n=0}^{\infty} A_n x^n
  \]
- Root difference \( r_1 - r_2 \) not an integer:
  \[
  y_2(x) = k y_1(x) \ln(x) + \sum_{n=0}^{\infty} A_n x^n
  \]
- Roots differ by integer (\( k \) may be 0).

**Bessel's Equation**

- Arises in mechanical and thermal problems in circular geometries.
- The value of \( \nu \) is a known parameter.
- Solve as example of Frobenius method.

\[
\begin{align*}
\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left( \frac{x^2 - \nu^2}{x^2} \right) y &= 0 \\
\frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y(x) &= x^\nu \sum_{n=0}^{\infty} a_n x^{n+\nu} = \sum_{n=0}^{\infty} a_n x^{n+\nu}
\end{align*}
\]

**Bessel's Equation II**

- Plug solution and derivatives into Bessel's equation and rearrange:

\[
\begin{align*}
\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \left( x^2 - \nu^2 \right) \sum_{n=0}^{\infty} a_n x^{n+\nu} &= 0 \\
\sum_{n=0}^{\infty} \left[ (n+r)^2 - \nu^2 \right] a_n x^{n+r} &= 0
\end{align*}
\]

**Bessel's Equation III**

- Final arrangement gets indicial equation:

\[
\begin{align*}
\sum_{n=0}^{\infty} (n+r)^2 - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} &= (0+r)^2 - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} \\
(1+r)^2 - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} &= \sum_{n=0}^{\infty} (n+r)^2 a_n x^{n+r} + \sum_{n=0}^{\infty} x^{-n-\nu} a_n x^{n+r} = 0
\end{align*}
\]

- Indicial equation \( (r^2 - \nu^2 = 0) \) roots \( r = \pm \nu \):
  - Solution gives double root if \( \nu = 0 \)
  - Roots differ by an integer for integer \( \nu \)
  - Roots do not differ by an integer for non-integer \( \nu \).
Bessel’s Equation IV

- Continue next week after midterm
- Get series solutions for Bessel functions for three cases
  - Double root for \( \nu = 0 \)
  - Roots differing by an integer
  - Non-integer roots
- Find two different series for any value of \( \nu \) just like finding sine and cosine series in power series solution for \( \frac{d^2y}{dx^2} + k^2y = 0 \)

Review for Midterm

- Vectors, matrices and determinants
  - Basic operations, particularly multiplication
  - Find determinants and matrix inverses
  - Vectors are linearly dependent if \( \sum \alpha_i v_i = 0 \) with at least one \( \alpha_i \neq 0 \)
  - A basis set for an n-dimensional vector space has n linearly independent vectors that can represent any vector in the space
- Gauss elimination process for solving equations determines linear dependence

Midterm Review II

- Solutions to linear equations \( Ax = b \)
  - Unique if \( \text{rank}[A \ b] = \text{rank} A = N_{\text{unknowns}} \)
  - Infinite solutions if \( \text{rank}[A \ b] = \text{rank} A \) is less than number of unknowns
  - No solution if \( \text{rank}[A \ b] \neq \text{rank} A \)
- Eigenvalues and eigenvectors: \( Ax = \lambda x \)
  - \( \text{Det}(A - \lambda I) = 0 \) gives eigenvalues
  - Solve \( (A - \lambda I)x = 0 \) for components of each eigenvector (one component arbitrary)

Midterm Review III

- Diagonalize a matrix: \( \Lambda = X^{-1}AX \)
  - \( X \) is matrix of eigenvectors
  - \( \Lambda \) is diagonal matrix of eigenvalues
  - Works only if \( X \) has an inverse
- Special matrices
  - Unitary matrix columns have \( [x^*(i)]x(j) = \delta_{ij} \)
  - Orthogonal matrix columns have \( [x^*(i)]x(j) = \delta_{ij} \)
  - Hermitian matrix \( A^* = A^T \)
    - For Hermitian matrix \( A^{-1} = A^T \)

Midterm Review IV

- First-order differential equations
  - Separable forms, e.g. \( \frac{dy}{dx} = f(x)g(y) \)
  - General linear equation \( \frac{dy}{dx} + f(x)y = g(x) \) has solution \( y = e^{-p}[C + e^{p}g(x)dx] \) where \( p = \int f(x)dx \)
  - Other separable forms
  - Solutions to \( dy/dx = f(x,y) \) exist over a region about \( x_0 < \min(a, b/K) \) where \( a, b \) are \( x, y \) borders and \( K = \max(f) \)
  - Unique solution if \( |f/|x| \) is bounded

Midterm Review V

- Second-order differential equations with constant coefficients: \( d^2y/dx^2 + \alpha dy/dx + \beta y = r(x) \): find \( \lambda_1 \) and \( \lambda_2 \)
  - \( \lambda_1 = -\alpha + \sqrt{\alpha^2 - 4\beta} \)
  - \( \lambda_2 = -\alpha - \sqrt{\alpha^2 - 4\beta} \)
  - \( r(x) = 0 \) gives homogenous solution, \( y_H \)
    - For real \( \lambda_1 \) and \( \lambda_2 \), \( y_H = C_1e^{\lambda_1} + C_2e^{\lambda_2} \)
    - For real \( \lambda_1 = \lambda_2 = \lambda \), \( y_H = (C_1 + C_2x)e^{\lambda x} \)
    - For complex roots, \( y = Acos(\omega x) + Bsin(\omega x) \), where \( \omega^2 = \beta - (\alpha/2)^2 = \beta - \alpha^2/4 \)
    - For \( r(x) \neq 0 \) \( y = y_H + y_P \)
Midterm Review VI

• For nonhomogeneous solutions find solution \( y = y_H + y_P \)
• To get particular solution, \( y_P \)
  – Write form for \( y_p \), based on form for \( r(x) \)
  – Substitute postulated \( y_p \) with unknown constant(s) into particular equation
  – Equate coefficients of like terms to find unknown constants
  – Use \( y = y_H + y_P \) to find constants from homogenous solution from boundary values

Midterm Exam

• Open book and notes, including homework solutions
• Make your own notes to use for exam
  – You are in trouble if you have to use the book on an open-book exam
• May be useful to have integral tables
• More credit given for showing how to obtain solution than for providing final details of algebra or arithmetic