Outline

- Review last class and homework
- Apply material from last class to mechanical vibrations
- Higher order equations with constant coefficients
  - Homogenous and nonhomogenous solutions
- Existence and uniqueness of solutions for higher order equations

Review Undetermined Coefficients

- Used for constant coefficient equation \( y'' + ay' + by = r(x) \)
- Solution is \( y = y_p + y_H \), where \( y_H \) is solution of \( y_H'' + ay_H' + by_H = 0 \)
- Postulate a solution for \( y_p \) following guidelines on next two charts
- Plug solution into ODE and solve for unknown coefficients
  - Overall coefficients of like terms on both sides of ODE must vanish

Table of Trial \( y_p \) Solutions

<table>
<thead>
<tr>
<th>For these ( r(x) )</th>
<th>Start with this ( y_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(x) = Ae^{ax} )</td>
<td>( y_p = Be^{ax} )</td>
</tr>
<tr>
<td>( r(x) = Ax^n )</td>
<td>( y_p = a_0 + a_1x + \ldots + a_nx^n )</td>
</tr>
<tr>
<td>( r(x) = Asin\omega t )</td>
<td>( y_p = B \sin\omega t + C \cos\omega t )</td>
</tr>
<tr>
<td>( r(x) = Acos\omega t )</td>
<td>( y_p = e^{ax} (B \sin\omega t + C \cos\omega t) )</td>
</tr>
</tbody>
</table>

Special Rules

- If the right-hand-side, \( r(x) \) consists of more than one term from the previous table, use a \( y_p \) that contains all the corresponding \( y_p \) terms
  - For \( r(x) = Acos bx + Ce^{ax} \), use \( y_p = E \sin bx + F \cos bx + Ge^{ax} \)
- If \( r(x) \) is proportional to a solution for the homogenous equation, use \( y_p \) equal to \( x \) times the \( y_p \) shown in the table
  - For a double root, multiply table \( y_p \) by \( x^2 \)

Review Parameter Variation

- Want to solve linear equation
  \[ \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x) \]
- Have to solve homogenous equation to get two (LI) solutions \( y_1 \) and \( y_2 \)
- Define \( W \) from these two solutions
  \[ W = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \]
Review Parameter Variation II

- Define \( u(x) \) and \( v(x) \) such that \( y_p = y_1u + y_2v \), where \( u \) and \( v \) are found from the following integrals
  \[
  u = -\int \frac{y_2r(x)}{W(x)} \, dx, \quad v = \int \frac{y_1r(x)}{W(x)} \, dx
  \]
  \[
  y_p = y_1u + y_2v = -y_1\int \frac{y_2r(x)}{W(x)} \, dx + y_2\int \frac{y_1r(x)}{W(x)} \, dx
  \]

- Get \( y = y_H + y_p \) and evaluate constants in \( y_H \) solution from initial conditions

Nonhomogenous Summary

- Undetermined coefficients is simpler approach but is limited
  - Constant coefficient equations
  - Limited set of functions
- Variation of parameters is more complex, but handles more cases
- In reality, there are no general methods to get homogenous solution to linear, second-order ODE without constant coefficients

Higher Order Equations

- General \( n^{th} \) order linear equation
  \[
  \frac{d^n y}{dx^n} + p_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_1(x) \frac{dy}{dx} + p_0(x)y = r(x)
  \]
- Treatment similar to second order
- Look at homogenous solution first
- Combine with particular solution
- Must consider ODE with constant coefficients to get any general results
  - This is similar to second order

Higher Order Equations II

- Look at general \( n^{th} \) order differential equation with constant coefficients
  \[
  \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0y = r(x)
  \]
- Find \( y_H \) from homogenous ODE
  \[
  \frac{d^n y_H}{dx^n} + a_{n-1} \frac{d^{n-1} y_H}{dx^{n-1}} + \cdots + a_1 \frac{dy_H}{dx} + a_0y_H = 0
  \]
- Homogenous solution
  \[
  y_H = \sum_{k=0}^{n} C_k e^{\lambda_k x}
  \]

Higher Order Equations III

- Homogenous solution
  \[
  y_H = \sum_{k=0}^{n} C_k e^{\lambda_k x}
  \]
- In homogenous solution, the values of \( \lambda_k \) are solutions to the equation \( \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda + a_0 = 0 \)
- Complex solutions occur as complex conjugates giving sines and cosines
- For double roots \( \lambda_k = DR \), we modify solution to give \( (C_k + C_{k+1}x) e^{(DR)x} \)

Higher Order Equations IV

- For nonhomogenous equations we can find the total solution \( y = y_H + y_p \)
- \( y_p \) may be found by undetermined coefficients or variation of parameters
  - Use same process for method of undetermined coefficients
  - Variation of parameters is more complex since it involves solution of simultaneous equations for new solutions
Higher Order Equations V

- There are \( n \) linearly-independent solutions to a linear, homogenous \( n \)th order ODE
- The \( n \) linearly-independent solutions form a basis for all solutions
  - Use same process for method of undetermined coefficients
  - Variation of parameters is more complex since it involves solution of simultaneous equations for new solutions

Existence and Uniqueness

- General linear, homogenous, \( n \)th order ODEs have a unique solution over \( a < x < b \) if all the \( p_k(x) \) are continuous there
  \[
  \frac{d^n y}{dx^n} + p_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_1(x) \frac{dy}{dx} + p_0(x)y = 0
  \]
- The proposed solutions \( y_k(x) \) to the homogenous ODE are linearly independent if the Wronskian (see next chart) is nonzero

Wronski Determinant

- Wronskian, \( W \), for \( n \)th order ODE (with notation that \( y^{(k)} \) denotes \( \frac{d^k y}{dx^k} \))

\[
W = \begin{vmatrix}
  y_1 & y_2 & y_3 & \cdots & y_n \\
  y_1^{(1)} & y_2^{(1)} & y_3^{(1)} & \cdots & y_n^{(1)} \\
  y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & y_n^{(2)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)}
\end{vmatrix}
\]

Application: Structural Member

- An elastic beam with an applied load, \( f(x) \), per unit length, in the \( y \) direction (normal to the beam)
  - Beam is bent under this load
  - Bending moment, \( M(x) \) is given by second-order ODE: \( \frac{d^2 M}{dx^2} = f(x) \)
  - Final deflection is \( \frac{d^2 y}{dx^2} \)
- \( M = EI \frac{d^2 y}{dx^2} \) where \( E \) is Young's modulus and \( I \) is moment of inertia

Structural Member ODE

- Combine \( \frac{d^2 M}{dx^2} = f(x) \) and \( M = EI \) \( \frac{d^2 y}{dx^2} \) to get \( EI \frac{d^4 y}{dx^4} = f(x) \)
  - SI units for these quantities are meters for \( x \) and \( y \), N/m² for \( E \), m⁴ for \( I \), N/m for \( f(x) \), and N·m for \( M \)
  - dimensions for \( n \)th order derivative are dimensions of numerator divided by (denominator dimensions)\(^n\)
  - Have a total of four boundary conditions at \( x = 0 \) and \( x = L \)
  - Equation has separable solution

Solving the Equation

\[
\frac{d^4 y}{dx^4} = \frac{f(x)}{EI} \quad \Rightarrow \quad \frac{d^4 y}{dx^4} = \frac{1}{EI} \int f(x)dx + C_1
\]
\[
\frac{d^3 y}{dx^3} = \frac{1}{EI} \int \left( \int f(x)dx \right) dx + C_2 + C_1x + C_3
\]
\[
\frac{d^2 y}{dx^2} = \frac{1}{EI} \int \left( \int \left( \int f(x)dx \right) dx \right) dx + C_3x^2 + C_2x + C_4 + C_1x + C_2
\]
\[
\frac{dy}{dx} = \frac{1}{EI} \int \left( \int \left( \int \left( \int f(x)dx \right) dx \right) dx \right) dx + C_4x^3 + C_3x^2 + C_2x + C_1x + C_3
\]

- Apply boundary conditions to find constants of integration
Application: Forced Vibrations

- Last week we showed solutions for free vibrations of spring-mass-damper system
- ODE was \( md''y/dt^2 + cdy/dt + ky = 0 \)
- Imposed force gives nonhomogenous ODE \( md''y/dt^2 + cdy/dt + ky = f(t) \)
- Consider example where \( f(t) = F_0 \cos \omega t \)
- Undetermined coefficient trial solution is \( y_P = A \sin \omega t + B \cos \omega t \)

Forced Vibrations II

- Derivatives of \( y_P = A \sin \omega t + B \cos \omega t \)
- \( y_P' = \omega A \cos \omega t - \omega B \sin \omega t \)
- \( y_P'' = -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t \)
- Substitute into ODE: \( m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F_0 \cos \omega t \)
- \( m \left( -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t \right) + c \left( \omega A \cos \omega t - \omega B \sin \omega t \right) + k \left( A \sin \omega t + B \cos \omega t \right) = F_0 \cos \omega t \)

Forced Vibrations III

- Rearrange to collect sines and cosines
- \( m \left( -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t \right) + c \left( \omega A \cos \omega t - \omega B \sin \omega t \right) + k \left( A \sin \omega t + B \cos \omega t \right) = F_0 \cos \omega t \)
- \( \left[ -m \omega^2 A - c \omega B + kA \right] \sin \omega t + \left[ -m \omega^2 B + c \omega A + kB \right] \cos \omega t = F_0 \cos \omega t \)
- Equate coefficients of sine and cosine terms on both sides of the equation

Define \( \omega_0^2 = k/m \) in Solution

- \( A = \begin{vmatrix} 0 & -\omega c \\ F_0 & k-m \omega^2 \end{vmatrix} \)
- \( B = \begin{vmatrix} k-m \omega^2 & 0 \\ \omega c & F_0 \end{vmatrix} \)
- \( A = \frac{F_0 \omega c}{(k-m \omega^2)^2 + \omega^2 c^2} \)
- \( B = \frac{F_0 (k-m \omega^2)}{(k-m \omega^2)^2 + \omega^2 c^2} \)
- \( A = \frac{m F_0 (\omega_0^2 - \omega^2)}{m^2 (\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \)
- \( B = \frac{m F_0 (\omega_0^2 - \omega^2)}{m^2 (\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \)

Forced Vibrations IV

- \( k \omega_A - c \omega B = 0 \) (sine terms)
- \( c \omega A - k \omega_B = F_0 \) (cosines)
- Cramer’s rule solution gives

Undamped Case, \( c = 0 \)

- \( y_P = A \sin \omega t + B \cos \omega t = B \cos \omega t \)
- From last week, \( y_H = C \sin \omega_0 t + D \cos \omega_0 t = E \cos (\omega_0 t + \delta) \)
- Look at initial conditions
Second and Higher Order Linear
Differential Equations

Undamped Case II

\[ y = y_D + y_P = C \sin \omega_0 t + D \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

- Initial conditions \( y(0) = y_0 \) and \( y'(0) = v_0 \)
- \( y_0 = D + F_0/m(\omega_0^2 - \omega^2) \)
- \( v_0 = \omega_0 C \)

\[ y = \frac{v_0}{\omega_0} \sin \omega_0 t + \left[ \frac{y_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}}{\omega_0} \right] \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

\[ \frac{y}{y_0} = \frac{v_0}{\omega_0 y_0} \sin \omega_0 t + \left[ 1 - \frac{\frac{F_0}{m(\omega_0^2 - \omega^2)}}{\omega_0} \right] \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

Undamped Case III

- Compute dimensionless \( C \) and \( \delta \)

\[ C = \sqrt{A^2 + B^2} = \left( \frac{\frac{y_0}{\omega_0} \sin \omega_0 t + \left[ \frac{y_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}}{\omega_0} \right] \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t}{y_0} \right)^2 \]

\[ \delta = \tan^{-1} \left( \frac{A}{B} \right) \]

\[ y = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

Undamped Case IV

- Start with solution below and convert \( \omega_0 t \) sine and cosine terms to a cosine term

\[ y = A \sin \omega_0 t + B \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

\[ A = \frac{y_0}{\omega_0} \quad B = y_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)} \]

\[ y = C \cos (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

\[ C = \sqrt{A^2 + B^2} = \left( \frac{\frac{\frac{y_0}{\omega_0} \sin \omega_0 t + \left[ \frac{y_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}}{\omega_0} \right] \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t}{y_0} \right)^2 \]

\[ \delta = \tan^{-1} \left( \frac{A}{B} \right) \]

\[ y = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

Undamped Case V

- Without forcing \( (F_0 = 0) \), when \( y_0 = v_0 = 0 \), the solution is \( y = 0 \) for all \( t \)
- Forcing gives a nonzero solution
- Start with general solution for \( c = 0 \)

\[ y = y_D + y_P = C \sin \omega_0 t + D \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

\[ v = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \left( \frac{\frac{F_0}{m(\omega_0^2 - \omega^2)}}{y_0} \right) \cos \omega t \]

\[ y_0 = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \left( \frac{\frac{F_0}{m(\omega_0^2 - \omega^2)}}{y_0} \right) \cos \omega t \]

Zero Initial Conditions

- Forcing gives a nonzero solution
- Start with general solution for \( c = 0 \)

\[ y = y_D + y_P = C \sin \omega_0 t + D \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \]

\[ v = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \left( \frac{\frac{F_0}{m(\omega_0^2 - \omega^2)}}{y_0} \right) \cos \omega t \]

\[ y_0 = \frac{C}{y_0} \cos (\omega_0 t - \delta) + \left( \frac{\frac{F_0}{m(\omega_0^2 - \omega^2)}}{y_0} \right) \cos \omega t \]

ME 501A Seminar in Engineering Analysis
Zero Initial Conditions II

- Rearrange solution for $c = y_0 = v_0 = 0$

$$ y = \frac{F_0}{m \omega^2 \left(1 - \omega^2/\omega_0^2\right)} \left[ \cos \left(\frac{\omega}{\omega_0} t \right) - \cos \omega t \right] $$

$$ \frac{ym \omega_0^2}{F_0} = \frac{ym k}{F_0} = \frac{yk}{F_0} = \frac{1}{1 - \omega^2/\omega_0^2} \left[ \cos \left(\frac{\omega}{\omega_0} t \right) - \cos \omega t \right] $$

- Plot $y_k/F_0$ versus $\omega t$ with $\omega/\omega_0$ as a parameter

Resonance Condition

- Current equation for $y/y_0$ has several terms with $1 - \omega^2/\omega_0^2$ in denominator
- Solution is not valid when $\omega = \omega_0$
- If $\omega = \omega_0$, $r(x) = F_0 \cos \omega t$ is proportional to homogenous equation solution
- Have to get new particular solution
- Use undetermined coefficients approach starting with $y_P = t \left[ A \sin \omega t + B \cos \omega t \right]$

$c = 0$ Resonance Solution II

- Remember $\omega = \omega_0 = (k/m)^{1/2}$ here
- Derivatives of $y_P = t \left[ A \sin \omega t + B \cos \omega t \right]$
- $y_P' = t \left[ -\omega^2 A \sin \omega t - \omega B \cos \omega t \right] + A \sin \omega t + B \cos \omega t$
- $y_P'' = t \left[ -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t \right] + 2 \omega A \cos \omega t - 2 \omega B \sin \omega t$
- Substitute into ODE for $c = 0$ and $m = k_0^2$: $m d^2y_p/dt^2 + ky_p = md^2y_p'/dt^2 + m \omega^2 y_P = F_0 \cos \omega t$

$c = 0$ Resonance Solution III

- $m \left[ -\omega^2 A \sin \omega t - \omega^2 B \cos \omega t \right] + m \left[ 2 \omega A \cos \omega t - 2 \omega B \sin \omega t \right] + m \omega^2 \left[ A \sin \omega t + B \cos \omega t \right] = F_0 \cos \omega t$
- After cancellations we have $m \left[ 2 \omega A \cos \omega t - 2 \omega B \sin \omega t \right] = F_0 \cos \omega t$
- This gives $B = 0$ and $A = F_0/2 m \omega$
- $y_p = \left[ F_0/2m\omega \right] t \sin \omega t$
- Particular solution increases without bound as $t$ increases

Examine Damping

- Damped oscillations without external force
  - derived (homogenous equation) solutions last week
  - Three cases: underdamping, critical damping, and underdamping
  - All cases show $y$ goes to zero as $t$ increases
  - Look at particular solution only to show effect of forced oscillations
  - Effects come from amplitude of oscillations
General Case for $c \neq 0$

- Convert $y_P = C \sin \omega t + D \cos \omega t = E \cos(\omega t - \delta)$ to examine amplitude
- $C^2 + D^2 = E^2$ and $\delta = \tan^{-1}(C/D)$
- Apply this to write $y_P = E \cos(\omega t - \delta)$

$$y_P = \frac{F_o \omega \sin \omega t}{m(\omega_0^2 - \omega^2) + \omega^2 c^2} + \frac{F_{oC} \omega_0 \cos \omega t}{m(\omega_0^2 - \omega^2) + \omega^2 c^2}$$

$$\delta = \tan^{-1}\left[\frac{F_{oC} \omega_0}{m(\omega_0^2 - \omega^2) + \omega^2 c^2}\right] = \tan^{-1}\left(\frac{\omega_0}{\omega}\right)$$

Amplitude of $y_P$ versus $\omega$

$$c^2 = 2m^2(\omega_0^2 - \omega^2) \quad \Rightarrow \quad \omega_0 = \frac{c^2}{2m}$$

- Maximum amplitude equation not valid if $c^2/2m > \omega_0^2 = k/m$
- Look at behavior of $y_P = C \cos(\omega t - \delta)$ by examining $C$ versus $\omega$
- Write dimensionless equation for $C$, which has dimensions of length

$$\frac{m\omega_0^2}{F_o} = \frac{m\omega_0^2}{F_o} \frac{F_o}{\sqrt{m(\omega_0^2 - \omega^2) + \omega^2 c^2}}$$

Find $\omega$ that Maximizes $C$

$$C = \left\{ \frac{F_{oC} \omega_0}{\sqrt{m(\omega_0^2 - \omega^2) + \omega^2 c^2}} \right\}^2 + \left\{ \frac{F_{oC} \omega_0}{\sqrt{m(\omega_0^2 - \omega^2) + \omega^2 c^2}} \right\}^2$$

$$= F_{oC} \omega_0 \sqrt{\frac{1}{m(\omega_0^2 - \omega^2) + \omega^2 c^2}}$$

$$dC = \frac{F_{oC} \omega_0}{\sqrt{m(\omega_0^2 - \omega^2) + \omega^2 c^2}}$$

$$c^2 = 2m^2(\omega_0^2 - \omega^2)$$

Amplitude of $y_P$ versus $\omega$ II

- Dimensionless amplitude depends on $\omega/\omega_0$ and $c^2/m^2 \omega_0^2 = c^2/mk$
- Previous result: $(\omega/\omega_0)^2_{\text{max}} = 1 - c^2/m^2 \omega_0^2$

Summary

- General solutions for ODEs with order $n \geq 2$ for constant coefficients only
  - Solutions are series of $e^{\lambda x}$ terms where $\lambda$ are solutions of algebraic equation
  - Special cases: double and complex roots
- Get general solution as $y = y_H + y_P$
  - Use method of undetermined coefficients (simpler than variation of parameters) to find $y_P$