Dimensionless Groups and Similitude

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Mechanical Engineering 390

Fluid Mechanics

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California State University Northridge

Outline

- Review momentum and energy balances
- Dimensionless analysis basics
- Dimensionless groups in fluid mechanics
  - Reynolds number
  - Drag coefficients
  - Others
- Experimental data and similitude

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Review Momentum

- Have balance equation for each component of momentum (k = x, y, z)
  - Use $V_x = u$, $V_y = v$, $V_z = w$, $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$
  - Speed = $V = (u^2 + v^2 + w^2)^{1/2}$

$$\frac{\partial (mV_k)_{cv}}{\partial t} + \sum_{o=1}^{N_{outlets}} \rho_o V_o A_o V_{k,o} - \sum_{i=1}^{N_{inlets}} \rho_i V_i A_i V_{k,i} = \sum F_k$$

$$\frac{\partial (mV_k)_{cv}}{\partial t} + \sum_{o=1}^{N_{outlets}} \dot{m}_o V_{k,o} - \sum_{i=1}^{N_{inlets}} \dot{m}_i V_{k,i} = \sum F_k$$

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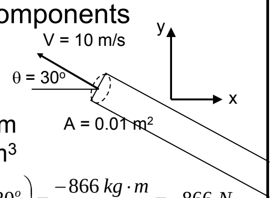
Review Momentum II

- Note differences in computation of mass flow and momentum components
- Example at right
  - $\dot{m} = \rho V A$
  - Find x and y momentum flows for ρ = 1000 kg/m³



$$\dot{m}u = \frac{1000\ kg}{m^3}\frac{10\ m}{s}\left(0.01\ m^2\right)\left(-\frac{10\ m}{s}\cos 30^o\right) = \frac{-866\ kg\cdot m}{s^2} = -866\ N$$

$$\dot{m}v = \frac{1000\ kg}{m^3}\frac{10\ m}{s}\left(0.01\ m^2\right)\left(\frac{10\ m}{s}\sin 30^o\right) = \frac{500\ kg\cdot m}{s^2} = 500\ N$$

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Review Energy Equation

$$E = \frac{mV^2}{2} + mgz + m\breve{u} \quad \Rightarrow \quad e = \frac{E}{m} = \frac{V^2}{2} + gz + \breve{u}$$

$$\frac{\partial E_{cv}}{\partial t} + \sum_{o=1}^{N_{outlets}} \rho_o V_o A_o e_o - \sum_{i=1}^{N_{inlets}} \rho_i V_i A_i e_i = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}}$$

$$\frac{\partial E_{cv}}{\partial t} + \sum_{o=1}^{N_{outlets}} \dot{m}_o e_o - \sum_{i=1}^{N_{inlets}} \dot{m}_i e_i = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}}$$

- Introduce shaft work and enthalpy, $\breve{h} = \breve{u} + P/\rho$

$$\frac{\partial E_{cv}}{\partial t} + \sum_{o=1}^{N_{outlets}} \dot{m}_o\left(\frac{V_o^2}{2} + gz_o + \breve{h}_o\right) - \sum_{i=1}^{N_{inlets}} \dot{m}_i\left(\frac{V_i^2}{2} + gz_i + \breve{h}_i\right) = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{shaft\\net\ in}}$$

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Review Energy and Bernoulli

- Steady energy equation with $\rho = \rho_o = \rho_i$ is Bernoulli equation with added terms

$$\frac{V_o^2}{2g} + \frac{P_o}{\gamma} + z_o = \frac{V_i^2}{2g} + \frac{P_i}{\gamma} + z_o + \frac{\dot{W}_{\substack{shaft\\net\ in}}}{\dot{m}g} - \frac{1}{g}\left(\breve{u}_o - \breve{u}_i - \frac{\dot{Q}_{\substack{net\\in}}}{\dot{m}}\right)$$

$$\frac{V_o^2}{2g} + \frac{P_o}{\gamma} + z_o = \frac{V_i^2}{2g} + \frac{P_i}{\gamma} + z_o + h_s - h_L$$

- Head loss, h_L is always positive
  - See example from last class at end of these notes

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Dimensionless Analysis

- Buckingham Pi Theorem
  - Can resolve a problem with N variables and D dimensions in to a set of N – D dimensionless variables
  - Part of chapter seven not assigned discusses general approach for doing this for general problems
  - Results for problems considered here are known and we will not cover this general approach

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Another Approach

- Write governing differential equations involving velocity components, u, v, w, pressure, and coordinates, x, y, z
- Define dimensionless variables, u/V_∞, v/V_∞, w/V_∞, x/L_{ref}, y/L_{ref}, z/L_{ref}, and $p/\rho V_\infty^2$
- Substitute dimensionless variables into differential equations and obtain quantities like Reynolds number ($Re = VL_{ref}\rho/\mu$) by rearrangement

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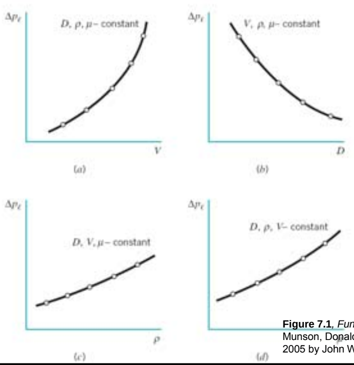
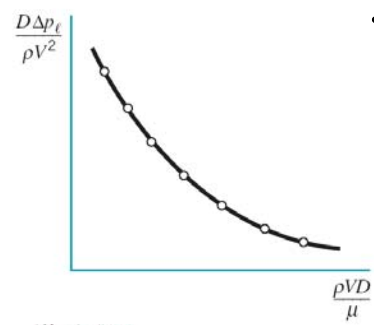
Example of Problem



- Expect pressure drop, $\Delta p_{loss}/L$ in smooth pipes to depend on variables D, ρ, μ, and V
  - 5 variables
  - 3 dimensions

Figure 7.1, *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi, 9

Example of Result



- Two dimensionless variables:
  - Reynolds number = ρVD/m
  - Loss parameter $D\Delta P/(L\rho V^2) = D\Delta P_\ell/(\rho V^2)$

California State University Northridge **Figure 7.2**, *Fundamentals of Fluid Mechanics, 5/E* by Bruce Munson, Donald Young, and Theodore Okiishi, 10

More Generally

- We would like to find the head loss in a pipe flow as a pressure head, Δp/γ
- We expect that this will include the following variables: V, L, D, m, and pipe roughness, ε
- In next chapter we will use the following relationship among dimensionless variables

$$\frac{(\Delta P)_{loss}}{\frac{\rho V^2}{2}}\frac{D}{L} = f\left(\frac{\rho VD}{\mu}, \frac{\varepsilon}{D}\right)$$

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Reynolds Number

- $Re = \rho VL_c/m$, where L_c is characteristic length for the geometry
  - L_c = D for pipe flow and external flows about cylinders and spheres
- Measures ratio of inertia (momentum) forces to viscous forces

$$\frac{momentum}{viscous} = \frac{(\rho VA)V}{\mu A\frac{dV}{dy}} \approx \frac{\rho V^2}{\mu\frac{V}{L_c}} = \frac{\rho VL_c}{\mu}$$

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Loss Coefficients

- Typically written as $\Delta P/(\rho V^2)$ or $\Delta P/(\rho V^2/2)$
- Measure of head loss due to viscous forces or form drag
- Drag coefficient C_D in external flows
  - D is drag force in N or lb_f
  - A is frontal area (projected area on plane perpendicular to the flow)
  - $C_D = (D/A) / (\rho V^2/2)$

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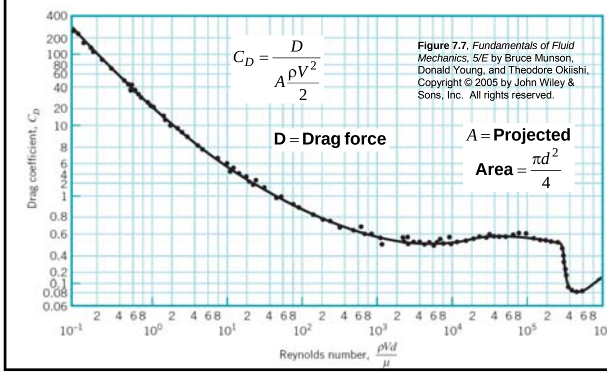
Example: Sphere Drag



Problem

- Estimate the drag force on a 1-cm-diameeter sphere in a 10-m/s air flow
  - Use air properties of air at atmospheric pressure and 300 K: ρ = 1,164 kg/m³ and μ = 184.6x10⁻⁷ N·s/m².

$$\mathrm{Re} = \frac{\rho V D}{\mu} = \frac{\frac{1.164\,kg}{m^3}\frac{10\,m}{s}(1cm)\frac{.01m}{cm}}{\frac{184.5x10^{-7}\,N\cdot s}{m^2}\frac{1kg\cdot m}{N\cdot s^2}} = 6{,}309$$

$C_D = 0.5$

$$D = C_D A\frac{\rho V^2}{2} = C_D\frac{\pi d^2}{4}\frac{\rho V^2}{2} = 0.5\frac{\pi(0.01m)^2}{4}\frac{1}{2}\frac{1.165\,kg}{m^3}\left(\frac{10\,m}{s}\right)^2\frac{1N\cdot s^2}{kg\cdot m} = 0.0023N$$

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More Complicated Problem

- Use the data on the drag ceofficient chart to find the terminal velocity of a 0.08 kg sphere with d = 0.01 m in air
  - Terminal velocity occurs when drag force equals force of gravity: $D = C_D A\rho V^2/2 = mg$
  - Use air properties for ρ and μ in trial and error solution
    - Guess V
    - Compute Re = $\rho Vd/\mu$, find C_D from chart, and compute $D = (\pi d^2/4)(\rho V^2/2)$
    - Continue until D is close enough to mg

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Other Dimensionless Variables

- Froude number, $Fr = V/(gL_c)^{1/2}$ is ratio of inertia force to gravitational force used in flows with a free surface
- Euler number, $Eu = p/\rho V^2$ is ratio of pressure force to inertia force, used in calculating pressure differences
- Cauchy number, $Ca = \rho V^2/E_v$ is ratio of inertia force to compressibility force used where compressibility important

$$E_T = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_T \qquad E_s = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s$$

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More Dimensionless Variables

- Mach number, Ma = V/c (c is sound speed) is square root of Cauchy number when E_s is used [$c = (E_s/\rho)^{1/2}$]
- Stroudahl number = $\omega L_c/V$, is ratio of two inertia forces: local oscillation with radian frequency ω and main flow
- Weber number, $We = \rho V^2 L_c/\sigma$, is ratio of inertia force to surface tension force

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Experimental Data

- Use dimensionless parameters to correlate experimental data
- Example MS student obtained data on pressure drop for 4 tube diameters and five different fluids (different ρ and μ)
- Initial data analysis showed lack of agreement between theory and data with no pattern in disagreement
- Replot as $f = (D\Delta p/L)/(\rho V^2)$ *vs* $Re = \rho Vd/\mu$

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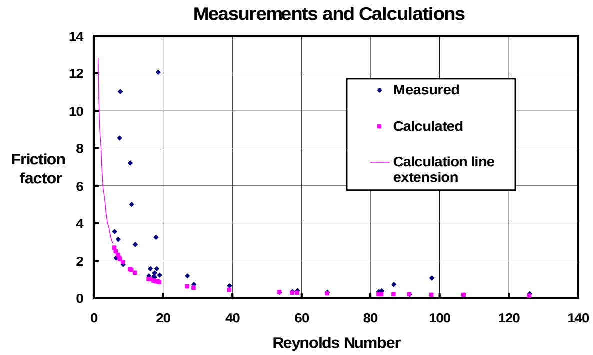

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Models

- Experiments on scale models in wind tunnels or towing tanks used to predict fluid flows and forces
- How are models and test procedures designed?
- Basic idea: keep all important dimensionless parameters the same between model and prototype (pre-production physical system to be built)

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Similitude

- Assume that we want to measure drag force in an incompressible flow by testing a model
- Drag forces depend on variables like velocity, density, viscosity, and size
- We saw results for C_D *vs*. Re for a sphere; expect similar results for model testing
- Want Re for model to be same as Re for prototype to get same C_D

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Typical Similarity Requirements

- For flow in full ducts and flows over external objects Reynolds number similarity is required
  - Forces found from by assuming model drag coefficient is same as prototype
- Open channel flows require Froude number similarity
  - Driving force is slope of channel captured by model scale

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Model Scales

- Given idea is that model size will be smaller than prototype
- Have length scale that applies to all dimensions
  - *E.g.* in a 1/50 scale model, all dimensions would 1/50th of the dimensions in the prototype
  - Look at effect that this has on parameters such are $Re = \rho VL_c/\mu$

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Model Scales II

- If we want $L_{c,model} = L_{c,prototype}/50$ with $Re_{model} = Re_{prototype}$, we must have

$$\frac{\rho_m V_m L_{c,m}}{\mu_m} = \frac{\rho_p V_p L_{c,p}}{\mu_p} \quad \Rightarrow \quad \frac{\rho_m V_m \mu_p}{\rho_p V_p \mu_m} = \frac{L_{c,p}}{L_{c,m}}$$

- If model test uses the same fluid as the prototype will use, we must have

$$\frac{V_m}{V_p} = \frac{L_{c,p}}{L_{c,m}}$$

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Model Scales III

- What are implications of this equation?

$$\frac{V_m}{V_p} = \frac{L_{c,p}}{L_{c,m}}$$

- If we have a small scale model, the test velocity for the model must be much greater than that for the model
- For flows in air the higher model velocity may have compressibility effects that are not present in the prototype

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Distorted Models

- Because of experimental difficulties in matching all dimensionless groups model testing is often done without only partial matching
- This necessary practice is an art and there is an extensive literature on such tests
- Open channel flows and building flows use distorted models

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Free Surface Models

- Froude number similarity important

$$Fr_m = \frac{V_m}{\sqrt{g_m L_{c,m}}} = \frac{V_p}{\sqrt{g_p L_{c,p}}} = Fr_p \quad \Rightarrow \quad \frac{V_m}{V_p} = \sqrt{\frac{L_{c,m}}{L_{c,p}}}$$

- For viscous effects want similar Reynolds number

$$\frac{\rho_m V_m L_{c,m}}{\mu_m} = \frac{\rho_p V_p L_{c,p}}{\mu_p} \quad \Rightarrow \quad \frac{\rho_m V_m \mu_p}{\rho_p V_p \mu_m} = \frac{L_{c,p}}{L_{c,m}}$$

$$\frac{L_{c,p}}{L_{c,m}} = \frac{\rho_m V_m \mu_p}{\rho_p V_p \mu_m} = \frac{\rho_m \mu_p}{\rho_p \mu_m}\sqrt{\frac{L_{c,m}}{L_{c,p}}} \quad \Rightarrow \quad \frac{\rho_m \mu_p}{\rho_p \mu_m} = \sqrt[3]{\frac{L_{c,p}}{L_{c,m}}}$$

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Free Surface Models II

- Last relation requires specific fluid property ratio that cannot easily be met
- For surface tension effects want similar Weber number
  - Same Weber and Froude numbers require

$$\frac{\rho_m V_m^2 L_{c,m}}{\sigma_m} = \frac{\rho_p V_p^2 L_{c,p}}{\sigma_p} \quad \Rightarrow \quad \frac{\rho_m V_m^2 \sigma_p}{\rho_p V_p^2 \sigma_m} = \frac{L_{c,p}}{L_{c,m}}$$

$$\frac{L_{c,p}}{L_{c,m}} = \frac{\rho_m \sigma_p}{\rho_p \sigma_m}\frac{V_m^2}{V_p^2} = \frac{\rho_m \sigma_p}{\rho_p \sigma_m}\frac{L_{c,m}}{L_{c,p}} \quad \Rightarrow \quad \frac{\rho_m \sigma_p}{\rho_p \sigma_m} = \left(\frac{L_{c,p}}{L_{c,m}}\right)^2$$

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Free Surface Models III

- For most free-surface models, the Froude number is the important similarity parameter
- Matching Reynolds number is less important
  - Exception is ship models
- Surface tension effects are usually not significant
- Open channel flow models often use distorted scales with vertical scale smaller than the horizontal scale

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Problem

- It is desired to test a model of a car with a maximum dimension of 20 ft in a wind tunnel that can accommodate a maximum length of 4 ft. What air velocity is required to test a prototype speed of (a) 20 mph, (b) 90 mph
- Reynolds number similarity is required

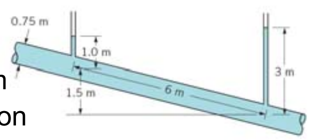
$$\frac{\rho_m V_m L_{c,m}}{\mu_m} = \frac{\rho_p V_p L_{c,p}}{\mu_p} \quad \Rightarrow \quad \frac{\rho_m V_m \mu_p}{\rho_p V_p \mu_m} = \frac{L_{c,p}}{L_{c,m}}$$

Problem Solution

- For same air properties in model and prototype $\frac{V_m}{V_p} = \frac{L_{c,p}}{L_{c,m}} = \frac{20\ ft}{4\ ft} = 5$
- Testing prototype speed of 20 mph requires model speed of 100 mph
- 90 mph for prototype requires 450 mph for model which is not valid because of compressibility effects

Energy Example and Solution

- ***From last class***
- **Given:** flow as shown in diagram
- **Find:** flow direction and head loss

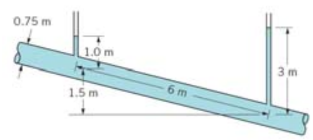


- Solve energy equation for head loss
- Head loss must be positive
  - Assume a flow direction and find head loss
    - If $h_L > 0$ assumption is correct and we know h_L
    - If $h_L < 0$ assumption is wrong, change h_L sign

Example Solution II

- Energy equation

$$\frac{V_o^2}{2g} + \frac{P_o}{\gamma} + z_o = \frac{V_i^2}{2g} + \frac{P_i}{\gamma} + z_i + h_s - h_L$$

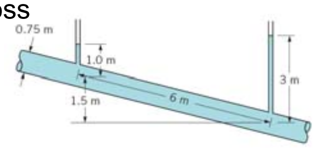


- Subscript "o" is outlet and "i" is inlet
- Assume flow is downhill (left to right)
  - No shaft work so $h_s = 0$
  - For incompressible fluid $V_i A_i = V_o A_o$ and with $A_i = A_o$, $V_i = V_o$, so V^2 terms cancel

Example Solution II

- Solve for head loss

$$h_L = \frac{p_i}{\gamma} + z_i - \left(\frac{p_o}{\gamma} + z_o\right)$$



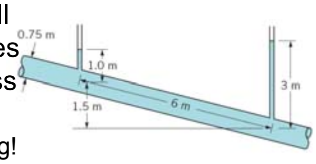
- Piezometer tubes measure z + p/γ

$$\frac{p_o}{\gamma} + z_o = 3\,m \qquad \frac{p_i}{\gamma} + z_i = 2.5\,m$$

$$h_L = \frac{p_i}{\gamma} + z_i - \left(\frac{p_o}{\gamma} + z_o\right) = 2.5\,m - 3\,m = -0.5\ \ m$$

Example Solution III

- Assumed downhill flow direction gives negative head loss
  - Original assumption wrong!



- Conclude that flow is uphill (from right to left) and correct h_L is 0.5 m (same magnitude as computed previously, but opposite sign).