Heat Transfer with Fins

Larry Caretto
Mechanical Engineering 375
Heat Transfer
February 21, 2007

Outline

- Review previous material
- What is a fin and why do we use them?
- Examples of fins
- Analysis of fins
- Fin effectiveness

Review Parallel Resistances

\[ \frac{1}{R_{\text{total}}} = \frac{1}{R_{\text{conv}}} + \frac{1}{R_{\text{rad}}} \]

Define total heat transfer coefficient, \( h_{\text{total}} \)

\[ Q = Q_{\text{conv}} + Q_{\text{rad}} \]

\[ h_{\text{total}} = \frac{1}{A R_{\text{total}}} = h_{\text{conv}} + h_{\text{rad}} \]

Review Slab with Convection

\[ \dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{h_1 L} \]

\[ T_1 = T_{\infty 1} \left( 1 - \frac{\dot{q}}{h_1} \right) \]

\[ T_2 = T_{\infty 2} + \frac{\dot{q}}{h_2} \]

Effect of Insulation Thickness

Review: Adding insulation to a cylinder (or sphere) can actually increase heat transfer!

Maximum heat transfer is at \( r_c = k/h \), which may be less than physical radius.
Resistances for Pipe Insulation

Review Conduction Shape Factor

- Simplified analysis
  - for multidimensional geometries with each surface at a uniform temperature
  - Use shape factor, $S$, whose equation is found from tables like Çengel Table 3-7
  - Basic equation: $\dot{Q} = kS(T_1 - T_2)$
  - $S$ must have dimensions of length
    - Equations for $S$ depend on parameters in the different geometries

What is a Fin?

- A fin is a extended surface to increase area for convection heat transfer
  \[ \dot{Q} = hA_s(T_s - T_\infty) \]
- Goal is to increase $A_s$ to increase $\dot{Q}$
  - Automobile radiator is example of fin

Innovative Fin Designs

- Important application is cooling of power electronics
- How do we analyze fin effectiveness?
  - Start with simple 1D fin

Analysis

- Conduction in x-direction
  - Convection from fin surface
  - Heat balance over differential volume element
    \[ -kA_s \frac{dT}{dx} = \left[ -kA_s \frac{dT}{dx} \right]_{x=0}^{x=\Delta x} + h\Delta x(T - T_\infty) \]

What is Fin Shape?

- Analysis does not consider shape of cross section, only area and perimeter
  - Derivation of fin equation assumes constant cross section
    - Fin may be in the shape of a uniform cylinder, but not a cone
- Is fin two dimensional?
  - Yes, but one-dimensional analysis is accurate if $h\delta/k < 0.2$, where $\delta = D$ for circular fins, thickness for rectangular fins
Fin Equation and Solution

- \( T_0 = T_b = \) base temperature at \( x = 0 \)
- Define \( \theta = T - T_\infty \) so \( \theta_b = T_b - T_\infty \)
- The fin equation for constant \( k \) and \( A_c \) is
  \[
  \frac{d^2 \theta}{dx^2} - m^2 \theta = 0 \quad m^2 = \frac{h_p}{kA_c}
  \]
- The solution to this differential equation is:
  \[
  \theta = C_1 e^{mx} + C_2 e^{-mx}
  \]

Boundary Conditions for \( C_1, C_2 \)

- First condition: base temperature is \( T_b \) so \( \theta = \theta_b = T_b - T_\infty \) at \( x = 0 \)
- Alternatives for second condition
  - Infinitely long fin (requires \( C_1 = 0 \))
  - Negligible heat transfer from end at \( x = L \)
  - Convection and radiation heat transfer coefficient for end at \( x = L \)

Infinitely Long Fin

- \( C_1 = 0 \) to keep solution finite
- At \( x = 0 \), \( \theta = \theta_b \) requires \( C_2 = \theta_b \)

\[
\theta_b e^{-mx} \Rightarrow T - T_\infty = (T_b - T_\infty) e^{-x\sqrt{hp/kA_c}}
\]

\[
\dot{q} = -k \frac{d\theta}{dx} = -k(T_b - T_\infty) e^{-x\sqrt{hp/kA_c}} (-\sqrt{hp/kA_c})
\]

\[
\dot{Q}_{x=0} = A_c \dot{q}_{x=0} = kA_c (T_b - T_\infty) \sqrt{hp/kA_c} = \sqrt{kA_c hp (T_b - T_\infty)}
\]

Zero Heat Transfer at \( x = L \)

- At \( x = 0 \), \( \theta = \theta_b \) at \( x = L \), \( d\theta/dx = 0 \)
- Algebraic details at end

\[
\theta = \theta_b \cosh m(L - x)
\]

\[
\dot{q} = -k \frac{d\theta}{dx} = -k\theta_b \frac{m \sinh m(L - x)}{\cosh mL} = k\theta_b m \sinh m(L - x)
\]

\[
\dot{Q}_{x=0} = A_c \dot{q}_{x=0} = kA_c \theta_b m \sinh m(L - x)
\]

\[
= \frac{kA_c hp (T_b - T_\infty) \tanh mL}{\cosh mL}
\]

Hyperbolic Tangent

- For convection (+ radiation) at \( x = L \) use approximation of extra length
- Use adjusted length, \( L_c \), to give area \( A_c \)
- Modified length, \( L_c = L + A_c/p \), will give convection from end
## Problem

- A pin (cylindrical) fin has a diameter of 4 mm, a length of 5 cm, and a thermal conductivity of 200 W/m·K. If the heat transfer coefficient is 70 W/m²·K, with a surface temperature of 50°C and an air temperature of 20°C, what is the heat transfer with and without the fin?  

  **Given:**  
  \( k = 200 \text{ W/m·K}, \ h = 70 \text{ W/m}^2\cdot\text{K}, \ D = 0.004 \text{ m}, \ L = 0.05 \text{ m}, \ T_s = 50^\circ\text{C}, \ T_\infty = 20^\circ\text{C} \),  

  **Find:** \( Q \) with and without fin

## Solution

- Use equation for finite length fin with end correction for convection

\[
\dot{Q}_{\text{fin}} = \sqrt{\frac{hA_c(T_b - T_\infty)\tanh mL_c}{\pi D/4}} = \sqrt{\frac{hA_c(T_b - T_\infty)}{\pi D^2/4}} = \sqrt{\frac{hA_c(T_b - T_\infty)}{0.01257 m^2}} = 18.71 m^2
\]

- Without fin available area is \( A_c \)

\[
\dot{Q}_{\text{no fin}} = hA_c(T_b - T_\infty) = 70 W / m^2 \cdot K 
\]

- Increase by factor of 39.64 (effectiveness, \( \varepsilon \))

## Increase of 40 times?

- Factor of 40 increase in heat transfer is just for area of fin
  
  - A practical installation of small fins like this one would have several fins on a surface
  
  - Consider two areas on original surface
    
    - Area where pins occur will have increase by factor shown in fin analysis
    
    - Area without fins has usual flat surface result

- Will consider this difference in determining fin effectiveness

## Fin Effectiveness

- Effectiveness, \( \varepsilon_{\text{fin}} \), is ratio of heat transfer with fin to heat transfer with no fin

\[
\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_c(T_b - T_\infty)}
\]

- \( A_c \) is fin area at base, same as \( A_c \) for fin with constant cross sectional area

- Want fin effectiveness greater than one to get additional heat transfer

## Constant \( A_c \), Fin Effectiveness

- Infinitely long fin

\[
\dot{Q}_{\text{fin}} = \sqrt{\frac{kA_c h(T_b - T_\infty)}{hA_c(T_b - T_\infty)\tanh mL}}
\]

- Fin with end convection

\[
\dot{Q}_{\text{fin}} = \sqrt{\frac{hA_c h(T_b - T_\infty)}{hA_c(T_b - T_\infty)\tanh mL}} = \sqrt{\frac{hA_c h(T_b - T_\infty)}{hA_c(T_b - T_\infty)\tanh mL}}
\]

\[
\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{hA_c(T_b - T_\infty)\tanh mL} = \frac{\dot{Q}_{\text{fin}}}{hA_c(T_b - T_\infty)\tanh mL}
\]

\[
\varepsilon_{\text{conv,fin}} = \frac{\dot{Q}_{\text{fin}}}{hA_c h(T_b - T_\infty)} = \frac{\dot{Q}_{\text{fin}}}{hA_c h(T_b - T_\infty)}
\]
What makes a fin effective?

$$\varepsilon = \frac{k \tan \theta}{h L}$$

- For infinite fin high $k$, low $h$ and high ratio of $p/A_c$ make a fin effective.
- For pin fin, $p = \pi D$ and $A_c = \pi D^2/4$, so $p/A_c = 4/D$; small $D$ gives better $\varepsilon$.
- Effect of variables on fin with convection at end is not clear.

What makes a fin effective? II

$$\varepsilon_{conv} = \frac{k \tan \theta}{h L}$$

- Choose a base case and vary each parameter from 0.1 to 10 times base value.
  - Rectangular fin with width, $w$, and thickness, $t$, so that $A_c = tw$ and $p = 2(t + w)$
- Base case has $k = 200 \text{ W/m} \cdot \text{K}$, $w = 0.03 \text{ m}$, $t = 0.005 \text{ m}$, $h = 70 \text{ W/m}^2 \cdot \text{K}$, $L = 0.08 \text{ m}$
  - Base $mL_c = 1.05$ and $\varepsilon = 28.5$

Effect of Parameters on Rectangular Fin Effectiveness

Overall Fin Effectiveness

- Original area, $A = (\text{area with fins, } A_{\text{fin}}) + (\text{area without fins, } A_{\text{unfin}})$

Fin Efficiency

- Compare actual heat transfer to ideal case where entire fin is at base temperature.

Fin Efficiency II

- Relation to effectiveness

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\varepsilon h A_c (T_b - T_\infty)}{h A_c (T_b - T_\infty)} = \frac{\varepsilon h A_c}{h A_c (T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_{\text{fin, max}}}(T_b - T_\infty)$$

- Recall pin fin problem with $D = 0.004 \text{ m}$ and $L_c = 0.051 \text{ m}$ so $A_{\text{pin}} = pL_c = \pi D L_c = \pi(0.004 \text{ m})(0.051 \text{ m}) = 0.0006409 \text{ m}^2$
  - Previously showed $\varepsilon = 39.64$ so $\eta = (39.64)/(pD^2/4)/(0.0006409 \text{ m}^2) = 0.777$
Electronic Heat Sinks

- Designed to protect equipment like power transistors from overheating
- Characterized by thermal resistance, $R$, such that $Q = \text{power dissipation} = \frac{(T_{\text{device}} - T_\infty)}{R}$
- See Table 3-6 in Çengel for examples with $R$ values
  - First part of table shown on next chart

Problem

- A power transistor that dissipates 120 W has a maximum operating temperature of 70°C. Cooling air is available at 25°C. Are any of the heat sinks on the previous page suitable for this transistor?
  - Given: $Q = 120$ W, $T_{\text{surf}} = 70$°C, $T_\infty = 25$°C
  - Find: Do devices have sufficient $R$ values

Solution

\[ Q = \frac{T_{\text{surf}} - T_\infty}{R} \Rightarrow R = \frac{T_{\text{surf}} - T_\infty}{Q} = \frac{70°C - 25°C}{120 W} = \frac{1.25°C}{W} \]

- Only the HS 5030 mounted vertically (with $R = 0.9°C/W$) will satisfy this cooling requirement
- The $R$ values for the other heat sinks are too large (1.2°C/W and 5°C/W)
Extra Charts

- The following charts show the details of the fin equation solution for no convection (dT/dx = 0) at x = L

Zero Heat Transfer at x = L

\[ \theta = C_1e^{mx} + C_2e^{-mx} \]

\[ \frac{d\theta}{dx} = \frac{dT}{dx} = \frac{d(T - T_b)}{dx} = mC_1e^{mx} - mC_2e^{-mx} \]

- At x = 0, \( \theta = \theta_b \); at x = L, \( \frac{d\theta}{dx} = \frac{dT}{dx} = 0 \)

\[ \theta_b = C_1 + C_2 \]

\[ \frac{d\theta}{dx}_{x=L} = \frac{dT}{dx}_{x=L} = mC_1e^{mL} - mC_2e^{-mL} = 0 \]

\[ \theta_b = C_1 + C_2e^{2mL} \]

\[ C_1 = \frac{\theta_b}{1 + e^{2mL}} \]

\[ C_2 = C_1e^{2mL} = \frac{\theta_be^{2mL}}{1 + e^{2mL}} \]

Zero Heat Transfer at x = L II

- Combine results from last chart

\[ \theta = C_1e^{mx} + C_2e^{-mx} \]

\[ C_1 = \frac{\theta_b}{1 + e^{2mL}} \]

\[ C_2 = \frac{\theta_be^{2mL}}{1 + e^{2mL}} \]

\[ \theta = \frac{\theta_b e^{mx} + \theta_be^{2mL}e^{-mx}}{1 + e^{2mL}} \]

- Multiply fraction top and bottom by \( e^{mL} \)

\[ \theta = \frac{\theta_b e^{mx} + \theta_be^{2mL}e^{-mx}}{1 + e^{2mL}} \]

\[ \theta = \frac{\theta_be^{mL}e^{mx} + \theta_be^{mL}e^{-mx}}{e^{mL} + e^{mL}} \]

Zero Heat Transfer at x = L III

- Rearrange exponential terms on last chart and introduce hyperbolic cosine, \( \cosh(x) = (e^x + e^{-x})/2 \)

\[ \theta = \frac{\theta_be^{-mL}e^{mx} + \theta_be^{mL}e^{-mx}}{e^{-mL} + e^{mL}} = \frac{\theta_be^{m(x-L)} + e^{-m(x-L)}}{2 \cosh mL} \]

\[ \cosh mL = \frac{2\theta_b \cosh m(L-x)}{\theta_b \cosh mL} \]

- Final result:

\[ \theta = \frac{\theta_b \cosh m(L-x)}{\cosh mL} \]