Outline

- Review last lecture
- Equivalent circuit analyses
  - Review basic concept
  - Application to series circuits with conduction and convection
  - Application to composite materials
  - Application to other geometries
- Two-dimensional shape factors

Review Steady, 1-D, $\dot{e}_{\text{gen}} = 0$

- Rectangular
  \[
  \dot{q} = \frac{\dot{Q}}{A} = -\frac{k(T_L - T_0)}{L}
  \]
- Cylindrical shell
  \[
  \dot{Q} = -\frac{2\pi k(T_L - T_1)}{\ln(r_2/r_1)}
  \]
- Spherical shell
  \[
  \dot{Q} = \frac{4\pi k(T_2 - T_1)}{1/r_1 - 1/r_2}
  \]
  $k$ is an average thermal conductivity (or a constant value) if $k$ is constant

$T_0, T_L$ = temperatures at $x = 0, L$; $T_1, T_2$ = temperatures at inner ($r_1$) and outer ($r_2$) radii

Review Heat Generation

- Various phenomena in solids can generate heat
- Define $\dot{e}_{\text{gen}}$ as the heat generated per unit volume per unit time

\[
\dot{e}_{\text{gen}} = \frac{i^2 R}{V} = \frac{A}{LA} = \frac{i^2 \rho L}{A^2}
\]

Review Heat Generation II

- Temperature and heat flux equations

\[
T = T_0 - \dot{e}_{\text{gen}} x^2 + \frac{\dot{e}_{\text{gen}} x L}{2k} \left(\frac{T_0 - T_L}{x}\right) L
\]

\[
\dot{q} = \frac{\dot{e}_{\text{gen}}}{2} + \frac{k(T_0 - T_L)}{L}
\]

$\dot{Q}_{x=0}$ + $\dot{E}_{\text{gen}} = \dot{Q}_{x=L}$
Steady Heat Transfer Definition

- In steady heat transfer the temperature and heat flux at any coordinate point do not change with time.
- Both temperature and heat transfer can change with spatial locations, but not with time.
- Steady energy balance (first law of thermodynamics) means that heat in plus heat generated equals heat out.

Rectangular Steady Conduction

The heat transfer is constant in this 1D rectangle for both constant & variable k.

\[
\dot{Q} = -k \frac{dT}{dx}
\]

Thermal Resistance

- Heat flow analogous to current.
- Temperature difference analogous to potential difference.
- Both follow Ohm's law with appropriate resistance term.

\[
\frac{\dot{Q}}{A} = -k \frac{dT}{dx}
\]

Thermal Resistance II

- Conduction
  \[
  \dot{Q} = \frac{kA(T_1 - T_2)}{L} \Rightarrow \dot{Q} = \frac{T_1 - T_2}{R_{\text{cond}}} \Rightarrow R_{\text{cond}} = \frac{L}{kA}
  \]
- Convection
  \[
  \dot{Q} = \frac{hA(T_s - T_f)}{R_{\text{conv}}} \Rightarrow \dot{Q} = \frac{T_s - T_f}{R_{\text{conv}}} \Rightarrow R_{\text{conv}} = \frac{1}{hA}
  \]
- Radiation
  \[
  R_{\text{rad}} = \frac{1}{A_1} \frac{1}{\sigma T_1^4 + T_2^4 + T_3^4 + T_4^4} = \frac{1}{A_1 h_{\text{rad}}}
  \]

Where Does the Heat Go?

Energy conservation requires that conduction heat through wall equals the heat leaving the wall by convection and radiation.

\[
\dot{Q}_1 = \dot{Q}_2 + \dot{Q}_3
\]
Parallel Resistances ($T_\infty = T_{\text{surf}}$)

\[ \frac{1}{R_{\text{total}}} = \frac{1}{R_{\text{conv}}} + \frac{1}{R_{\text{rad}}} \]

\[ R_{\text{total}} = \frac{1}{A_1 R_{\text{total}} = h_{\text{conv}} + h_{\text{rad}}} \]

Define total heat transfer coefficient, $h_{\text{total}}$

\[ h_{\text{total}} = \frac{1}{A_1 R_{\text{total}}} = h_{\text{conv}} + h_{\text{rad}} \]

Combined Modes

\[ \dot{q} = h(T_{\infty 1} - T_1) \]

Convection or convection plus radiation

\[ \dot{q} = \frac{k(T_1 - T_2)}{L} \]

Combined Modes II

\[ \dot{Q} = T_{\infty 1} - T_{\infty 2} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} \]

\[ \dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{conv}} + R_{\text{wall}} + R_{\text{conv}}.} \]

\[ \dot{q} = \frac{\dot{Q}}{A} = \frac{T_{\infty 1} - T_{\infty 2}}{L + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{k}} \]

Combined Modes III

\[ \dot{q} = \frac{\dot{Q}}{A} = \frac{T_{\infty 1} - T_{\infty 2}}{h_1 + \frac{1}{h_2} + \frac{1}{k}} \]

\[ \dot{q} = \frac{T_{\infty 1} - T_{\infty 2}}{h_1 + \frac{1}{h_2} + \frac{1}{k}} \]

If you know $h_1$, $h_2$, $L$, $k$, $T_{\infty 1}$, and $T_{\infty 2}$, but you do not know $T_1$ and $T_2$, can you find the heat flux?

Once you found the heat flux from the information give, can you find $T_1$ and $T_2$?

Problem

A house has a 4 in thick brick wall with $k = 0.6$ Btu/hr·ft·°F. The interior temperature is 70°F and the exterior temperature is 0°F. The inside and outside convection plus radiation coefficients are 3 Btu/hr·ft²·°F and 4 Btu/hr·ft²·°F, respectively.

Find the heat flux through the wall.

Given: Wall with $L = 4$ in = 4/12 ft and $k = 0.6$ Btu/hr·ft·°F has convection on two sides. $T_{\infty 1} = 70°F$, $T_{\infty 2} = 0°F$, $h_1 = 3$ Btu/hr·ft²·°F and $h_2 = 4$ Btu/hr·ft²·°F.

Find: $\dot{q} = \frac{\dot{Q}}{A}$
Solution II

A is area normal to heat flow

\[ \dot{q} = \frac{T_{a1} - T_1}{h_1} \implies T_1 = T_{a1} - \frac{\dot{q}}{h_1} = 70^\circ F - \frac{61.5 \text{ Btu}}{3 \text{ Btu}} = 49.5^\circ F \]

\[ \dot{q} = \frac{T_2 - T_{a2}}{1} \implies T_2 = T_{a2} + \frac{\dot{q}}{h_1} = 0^\circ F - \frac{61.5 \text{ Btu}}{4 \text{ Btu}} = 15.4^\circ F \]

Figure 3-6 from Çengel, Heat and Mass Transfer

Solution III

A is area normal to heat flow

\[ \dot{q} = \frac{Q}{A} = \frac{k(T_1 - T_2)}{L} = \frac{4}{15} \frac{\text{Btu}}{\text{hr} \cdot \text{ft}^2} \]

How can we check results below from analysis of overall problem and convection processes?

\[ \dot{q} = \frac{61.5 \text{ Btu}}{\text{hr} \cdot \text{ft}^2} = \frac{T_1}{49.5^\circ F} + \frac{T_2}{15.4^\circ F} \]

Analyze conduction step for consistency.

Composite Materials

How would you analyze this problem?

Review Cylindrical Shell

For constant k

\[ \dot{Q}_f = \frac{2\pi k (T_1 - T_2)}{L} \]

\[ R = \frac{1}{2\pi k} \ln \left( \frac{r_2}{r_1} \right) \]

\[ \dot{Q}_f = \frac{T_1 - T_2}{R} \]

Cylindrical Shell with Convection

\[ A_1 = 2\pi r L, \quad A_2 = 2\pi r L, \quad A_3 = \frac{1}{h_2} \ln \left( \frac{r_2}{r_1} \right) \]

\[ Q = \frac{T_{a1} - T_{a2}}{R_{\text{conv,1}} + R_{\text{conv,2}}} \]

\[ R_{\text{conv,1}} = \frac{1}{h_1 A_1} = \frac{1}{h_2} \frac{1}{2\pi r L}, \quad R_{\text{conv,2}} = \frac{1}{h_3 A_2} = \frac{1}{h_2} \frac{1}{2\pi r L} \]

Figure 3-25 from Çengel, Heat and Mass Transfer
Cylinder plus Convection Result

\[ Q = \frac{1}{h_1 2\pi r_1 L} + \frac{1}{2kL} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{h_2 2\pi r_2 L} \]

We can rearrange this equation as shown below

\[ \frac{Q}{L} = \frac{2\pi(T_{x1} - T_{x2})}{\frac{1}{h_1} + \frac{1}{k} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{h_2}} \]

Problem

A hot-water pipe (k = 35 Btu/hr·ft·°F) in a house, made of ¾ inch schedule 40 pipe (OD = 1,050 in; ID = 0.824 in) is 40 ft long and contains water at 120°F. The air around the pipe is at 60°F. The heat transfer coefficients inside and outside the pipe are, respectively, 200 and 3 Btu/hr·ft²·°F. Determine the heat loss from the pipe.

Solution

Given: \( T_{x2} = 60°F, T_{x1} = 120°F, r_1 = ID/2 = 0.412 \text{ in}, r_2 = OD/2 = 0.525 \text{ in}, k = 35 \text{ Btu/hr·ft·°F}, L = 40 \text{ ft}, h_1 = 200 \text{ Btu/hr·ft²·°F}, h_2 = 3 \text{ Btu/hr·ft²·°F} \)

Find: \( Q \)

\[ \frac{Q}{L} = \frac{2\pi(T_{x1} - T_{x2})}{\frac{1}{h_1} + \frac{1}{k} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{h_2}} \]

Composite Cylindrical Shell

\[ \frac{1}{h_{12}} = \frac{1}{h_1 2\pi r_1 L} + \frac{1}{h_2 2\pi r_2 L} \]

Composite Cylindrical Shell II
Another Problem

- Insulation with $k = 0.2\ \text{Btu/hr-ft-°F}$ is to be added to the pipe in the previous example problem. Determine the heat transfer if the insulation is one inch thick.

$$\dot{Q} = \frac{1}{h_1 2\pi r_L L} + \frac{1}{2\pi k_2 L} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{2\pi k_3 L} \ln \left( \frac{r_3}{r_2} \right) + \frac{1}{h_2 2\pi r_L L}$$

Know all terms from previous example except these two.

Another Problem II

Unchanged resistances from previous example

$$\dot{Q} = \frac{1}{h_1 2\pi r_L L} + \frac{1}{2\pi k_1 L} \ln \left( \frac{r_2}{r_1} \right) = \frac{0.153\ \text{hr} \cdot \text{ft}^2\text{°F}}{\text{Btu}}$$

New and modified resistances

$$\frac{1}{h_2 r_3} = \frac{h_2}{k_2} \frac{2.623\ \text{hr} \cdot \text{ft}^2\text{°F}}{3\ \text{Btu}} = \frac{12\ \text{in}}{1.525\ \text{in} / \text{ft}} = 8.02\ \text{Btu}$$

Another Problem III

$$\dot{Q} = \frac{2\pi L (T_{in} - T_{out})}{\frac{1}{h_1 r_1} + \frac{1}{k_1} \ln \left( \frac{r_2}{r_1} \right) + \frac{1}{k_2} \ln \left( \frac{r_3}{r_2} \right) + \frac{1}{h_2 r_3}$$

$$= \frac{1860\ \text{Btu}}{160\ \text{Btu}}$$

- Insulation and outer convection resistances are largest
  - Inner convection and pipe conduction negligible
  - Outer convection resistance less with insulation

Effect of Insulation Thickness

Insulation Increases $\dot{Q}$?

- Why does initial amount of insulation increase heat transfer?
  - Tradeoff of two resistances
  - Added insulation adds conduction resistance
  - Added insulation also increases outer radius which decreases the outer convection resistance $1/(h_{out}A_{out}) = 1/(h_{out}2\pi r_{out} L)$
Resistances for Pipe Insulation

Radius for Maximum $\dot{Q}$

- $r_o = k_o/h_2$ for maximum $\dot{Q}$
- In the example problem $h_2 = 3$ Btu/hr-ft$^2$-°F, and $k_o = 0.2$ Btu/hr-ft-°F so $r_o = 0.0667$ ft = 0.8 in for maximum $\dot{Q}$
- Pipe radius was 0.525 in; $r_o = 0.8$ in gives an insulation thickness of 0.275 in
- Note that $r_o = k_o/h_2$ does not depend on $r_i$ and is usually larger than $r_i$
- There is no radius for minimum $\dot{Q}$

Spherical Shell with Convection

Radius for Maximum $\dot{Q}$

- $r_o = k_o/h_2$ for maximum $\dot{Q}$
- In the example problem $h_2 = 3$ Btu/hr-ft$^2$-°F, and $k_o = 0.2$ Btu/hr-ft-°F so $r_o = 0.0667$ ft = 0.8 in for maximum $\dot{Q}$
- Pipe radius was 0.525 in; $r_o = 0.8$ in gives an insulation thickness of 0.275 in
- Note that $r_o = k_o/h_2$ does not depend on $r_i$ and is usually larger than $r_i$
- There is no radius for minimum $\dot{Q}$

Spherical Shell Result

Conduction Shape Factors

- Simplified analysis
  - for multidimensional geometries with each surface at a uniform temperature
  - Use shape factor, $S$, whose equation is found from tables like Çengel Table 3-7
  - Basic equation: $\dot{Q} = kS(T_1 - T_2)$
  - $S$ must have dimensions of length
  - Equations for $S$ depend on parameters in the different geometries
Example Shape Factor

(1) Isothermal cylinder of length $L$
buried in a semi-infinite medium
($L \gg D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$

From Table 7-1 in Çengel, Heat and Mass Transfer.

Buried Pipe Shape Factor

![Graph showing the shape factor for a buried pipe.](image-url)