Heat Transfer Basics

Larry Caretto
Mechanical Engineering 375

Heat Transfer

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Engineering Accreditation

- CSUN has accredited programs in Civil, Electrical, Manufacturing and Mechanical Engineering
  - National accrediting agency reviews all engineering programs in US
- Fall 2007 reaccreditation visit requires collection of student work
  - Turn in all your notes, quizzes design project and exams at end of semester
    - You can get them back in late fall 2007

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Assessment Results

- 19 students completed assessment
  - 17 are OK or better with Excel skills
  - 5 are OK or better with Matlab skills
- Course completion data: Math 280(14), ECE 240(12), ME 309(5), ME 370(9), ME 390(4), ME 470(0), MSE 304(12)
- 15 got $\int x^3 dx = x^4/4 + C$ (6 missed C)
- 14 got $d(e^{ax}) = e^{ax}$

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Assessment Results II

- 3 got thermo problem correct
  - $Q = Q/m = \int c_p dT$ for constant P
    - With constant c_p, $q = c_p(T_2 - T_1)$
- 5 got interpolation (4 got partial credit)

$$x = x_1 + \frac{x_2 - x_1}{y_2 - y_1}(y - y_1)$$

- 3 got problem to find Δh for $dh/dT = 1 + 100/T$ from T = 500 to T = 1000 (8 got partial credit)

$$h_2 - h_1 = \int dh = \int_{T_1}^{T_2}\left(1 + \frac{100}{T}\right)dT = \left[T + 100\ln T\right]_{T_1}^{T_2} = T_2 - T_1 + 100\ln\frac{T_2}{T_1}$$

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Outline

- Review last week
- Heat generation
- General energy balance and geometry
- Simplified cases: steady, one-dimensional, no heat generation, constant thermal conductivity
- Analyze one dimensional cases
  - Constant and variable thermal conductivity
  - Constant heat generation term

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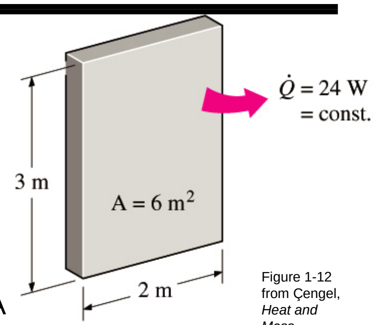
Review Definitions

Q is the total heat transfer with energy units of J or Btu

$\dot{Q}$ is the heat transfer rate in power units J/s = W or Btu/hr

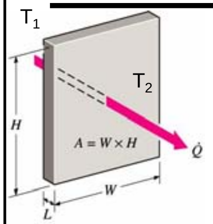
Heat flux: $\dot{q} = \dot{Q}/A$



$$\dot{q} = \frac{\dot{Q}}{A} = \frac{24\text{ W}}{6\text{ m}^2} = 4\text{ W/m}^2$$

Figure 1-12 from Çengel, *Heat and Mass Transfer*

Review Conduction Fourier Law



- $\dot{q}_x = -k\partial T/\partial x$ (1D: $-kdT/dx$)
- k is thermal conductivity (W/m·K or Btu/hr·ft²·°F)
  - k depends on temperature; may be assumed constant for small temperature range
- For constant k

$$\dot{q} = \frac{k(T_1 - T_2)}{L} \quad or \quad \dot{Q} = \dot{q}A = \frac{kA(T_1 - T_2)}{L}$$

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Review Convection

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

h = heat transfer coefficient (W/m²·K) or Btu/hr·ft²·°F

Equation assumes direction of heat transfer is from solid to fluid

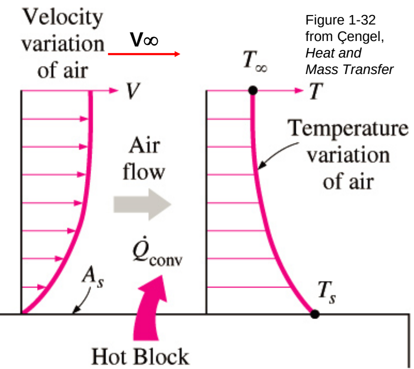

Figure 1-32 from Çengel, *Heat and Mass Transfer*

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Review Radiation

- Radiation from surface 1 to surface 2

$$\dot{Q}_{rad,1\to 2} = A_1\mathfrak{F}_{12}\sigma\left(T_1^4 - T_2^4\right)$$

  - $\mathfrak{F}_{12}$ is shape-emissivity factor
  - σ, Stefan Boltzmann constant = 5.670×10^{-8} W/m²·K⁴ = 0.1714×10^{-8} Btu/hr·ft²·R⁴
  - T is the absolute temperature!!!
  - Black body is perfect radiator
    - Emissivity is fraction of black body emitted by actual surface
    - Absorbtivity is incoming fraction absorbed

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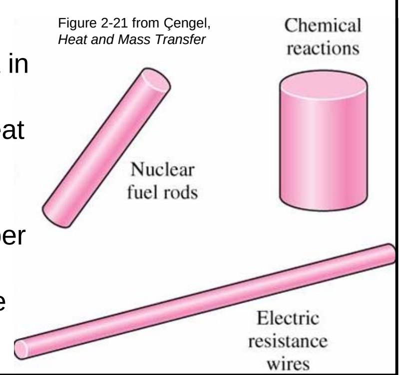
Heat Generation

- Various phenomena in solids can generate heat
- Define $\dot{e}_{gen}$ as the heat generated per unit volume per unit time

Figure 2-21 from Çengel, *Heat and Mass Transfer*



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Find $\dot{e}_{gen}$ for a Wire with Current

- The definition of $\dot{e}_{gen}$ is the heat generated per unit volume per unit time
  - Electrical resistance produces a heat dissipation of $I^2R = I^2\rho L/A$ in watts where
    - I is the current in amps
    - ρ is the electrical resistivity (ohm·m)
    - L is the length of the wire in m
    - A is cross sectional area of the wire, πr^2, in m²
  - Find an equation for $\dot{e}_{gen}$ in terms of the variables shown here

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Find $\dot{e}_{gen}$ for a Wire with Current

- The definition of $\dot{e}_{gen}$ is the heat energy generated per unit volume per unit time
  - Electrical resistance produces an energy dissipation of $I^2R = I^2\rho L/A$ in watts which is energy per unit time
    - Divide this by the wire volume, V = LA to get $\dot{e}_{gen}$

$$\dot{e}_{gen} = \frac{I^2R}{V} = \frac{\frac{I^2\rho L}{A}}{LA} = \frac{I^2\rho}{A^2} = \rho J^2 = \frac{I^2\rho}{\pi^2 r^4}$$

J = current density (A/m²) 12

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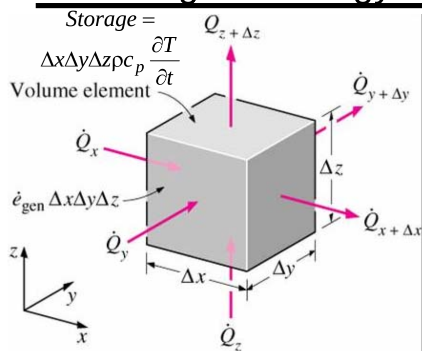
Find $\dot{e}_{gen}$ for a Wire with Current

- Apply the equation just found to find $\dot{e}_{gen}$ for a copper wire (ρ = 1.72x10^{-8} ohm·m at 20ºC) with a diameter of 1 mm (= 0.001 m) and a current of 10 amperes

$$\dot{e}_{gen} = \frac{I^2\rho}{A^2} = \frac{I^2\rho}{\left(\frac{\pi}{4}D^2\right)^2} = \frac{(10\ A)^2\left(1.72x10^{-8}\ ohm \cdot m\right)}{\left(\frac{\pi}{4}(0.001\ m)^2\right)^2 \frac{A^2 \cdot ohm}{W}} = \frac{2.788x10^6\ W}{m^3}$$

Rectangular Energy Balance



Energy balance: Stored energy = heat inflow – heat outflow + heat generated

Figure 2-21 from Çengel, *Heat and Mass Transfer*

Rectangular Energy Balance

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} + \dot{e}_{gen}$$

Stored energy = heat inflow – heat outflow + heat generated

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} + \dot{e}_{gen}$$

Uses Fourier Law $\dot{q}_\xi = -k\frac{\partial T}{\partial \xi}$

Energy Balance Dimensions

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} + \dot{e}_{gen}$$

$\rho c_p \frac{\partial T}{\partial t}$ dimensions: $\frac{M}{L^3}\frac{E}{M \cdot \Theta}\frac{\Theta}{T} = \frac{E}{L^3 \cdot T}$

E denotes energy units

$\frac{\partial}{\partial \xi}k\frac{\partial T}{\partial \xi}$ dimensions: $\frac{1}{L}\frac{E}{T \cdot L \cdot \Theta}\frac{\Theta}{L} = \frac{E}{L^3 \cdot T}$

All terms have dimensions of energy per unit volume per unit time

Energy Balance Simplifications

$$\cancel{\rho c_p \frac{\partial T}{\partial t}} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} + \dot{e}_{gen}$$

0 for steady heat transfer

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k\frac{\partial T}{\partial z} + \cancel{\dot{e}_{gen}}$$

0 for no heat generation

Energy Balance Simplifications

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \cancel{\frac{\partial}{\partial y}k\frac{\partial T}{\partial y}} + \cancel{\frac{\partial}{\partial z}k\frac{\partial T}{\partial z}} + \dot{e}_{gen}$$

0 for one dimensional heat transfer

For constant thermal conductivity, k

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{e}_{gen}$$

Simplest Cases

$$\rho c_p \cancel{\frac{\partial T}{\partial t}} = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} + \cancel{\frac{\partial}{\partial y} k \frac{\partial T}{\partial y}} + \cancel{\frac{\partial}{\partial z} k \frac{\partial T}{\partial z}} + \cancel{\dot{e}_{gen}}$$

0 for steady heat transfer

0 for one dimensional heat transfer

0 for no heat generation

$$\frac{d}{dx} k \frac{dT}{dx} = -\frac{d\dot{q}_x}{dx} = 0 \quad \Rightarrow \quad \dot{q}_x = C = \text{Constant}$$

$$\dot{q}_x = C = -k \frac{dT}{dx} \quad \Rightarrow \quad \int_0^L \dot{q}_x dx = \dot{q}_x L = -\int_{T_0}^{T_L} k dT$$

For constant k: $\dot{q}_x = k(T_0 - T_L)/L$

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Cylindrical Coordinates

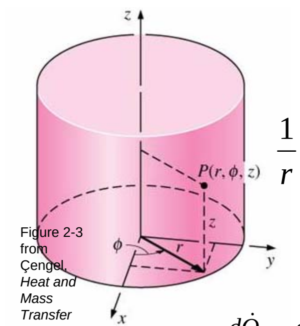

Figure 2-3 from Çengel, *Heat and Mass Transfer*

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} kr \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi} k \frac{\partial T}{\partial \phi} + \frac{\partial}{\partial z} k \frac{\partial T}{\partial z} + \dot{e}_{gen}$$

$$d\dot{Q}_r = \dot{q}_r dA = -k \frac{\partial T}{\partial r} r d\phi dz$$

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Steady 1-D Cylinder

$$\cancel{\rho c_p \frac{\partial T}{\partial t}} = \frac{1}{r} \frac{\partial}{\partial r} kr \frac{\partial T}{\partial r} + \cancel{\frac{1}{r^2} \frac{\partial}{\partial \phi} k \frac{\partial T}{\partial \phi}} + \cancel{\frac{\partial}{\partial z} k \frac{\partial T}{\partial z}} + \dot{e}_{gen}$$

$$0 = \frac{1}{r} \frac{d}{dr} kr \frac{dT}{dr} + \dot{e}_{gen}$$

For no heat generation

$$0 = \frac{d}{dr} kr \frac{dT}{dr} \quad \Rightarrow \quad kr \frac{dT}{dr} = C$$

Cylindrical radial heat transfer

$$\dot{Q}_r = \dot{q}_r A = -k \frac{\partial T}{\partial r} 2\pi r L = -2\pi L C$$

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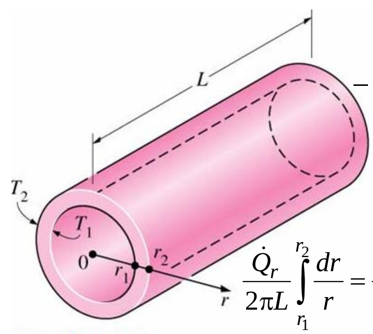
Cylindrical Shell



$$\dot{Q}_r = \dot{q}_r A = -k \frac{dT}{dr} 2\pi r L = -2\pi L C$$

$$\frac{\dot{Q}_r}{2\pi L} = -C = -kr \frac{dT}{dr}$$

$$\frac{\dot{Q}_r}{2\pi L} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\dot{Q}_r}{2\pi L} \ln\left(\frac{r_2}{r_1}\right) = -\int_{T_1}^{T_2} k dT$$

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Figure 2-50 from Çengel, *Heat and Mass Transfer*

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Cylindrical Shell II

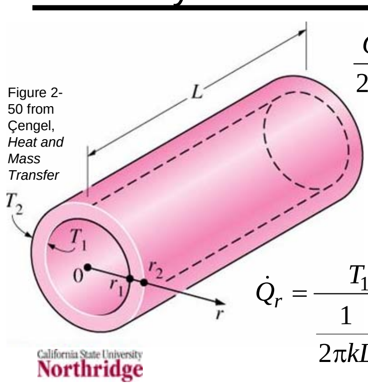

Figure 2-50 from Çengel, *Heat and Mass Transfer*

$$\frac{\dot{Q}_r}{2\pi L} \ln\left(\frac{r_2}{r_1}\right) = -\int_{T_1}^{T_2} k dT$$

For constant k

$$\frac{\dot{Q}_r}{L} = \frac{2\pi k (T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

$$\dot{Q}_r = \frac{T_1 - T_2}{\frac{1}{2\pi k L} \ln\left(\frac{r_2}{r_1}\right)}$$

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Sample Problem

$$\dot{Q}_r = \frac{T_1 - T_2}{\frac{1}{2\pi k L} \ln\left(\frac{r_2}{r_1}\right)}$$

lation (k = 0.04 W/m·K) is to be added to a pipe with a 0.15 m diameter, a surface temperature of 120ºC, and a heat loss per unit length of 25 W/m. What thickness of insulation, δ, is required if the temperature of the outer insulation surface is 40ºC?

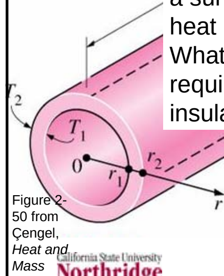

Figure 2-50 from Çengel, *Heat and Mass Transfer*

Given: T_1 = 120ºC and T_2 = 40ºC $\frac{\dot{Q}_r}{L} = \frac{25\ W}{m}$

r_1 = (0.15 m) / 2 = 0.075 m, and k

Find: δ = $r_2 - r_1$

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Sample Problem Solution

- **Given:** $T_1 = 120^oC$, $T_2 = 40^oC$, $r_1 = 0.075$ m, $k = 0.04$ W/m·K and $\dot{Q}_r/L = 25$ W/m

$$\dot{Q}_r = \frac{T_1 - T_2}{\frac{1}{2\pi k L}\ln\left(\frac{r_2}{r_1}\right)} \Rightarrow \ln\left(\frac{r_2}{r_1}\right) = 2\pi k \frac{T_1 - T_2}{\frac{\dot{Q}_r}{L}}$$

$$= 2\pi \frac{0.04\,W}{m \cdot {}^oC}\frac{120^oC - 40^oC}{\frac{25\,W}{m}} = 0.804 \Rightarrow r_2 = r_1 e^{0.804} = (0.075\,m)e^{0.804}$$

$$= 0.1676\,m \Rightarrow \delta = r_2 - r_1 = 0.1676\,m - 0.075\,m = 0.0926\,m$$

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Spherical Coordinates

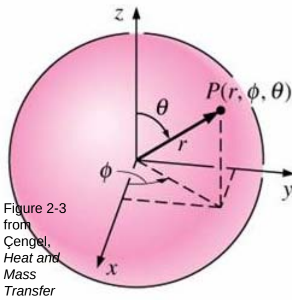

Figure 2-3 from Çengel, *Heat and Mass Transfer*

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}kr^2\frac{\partial T}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}k\sin\theta\frac{\partial T}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}k\frac{\partial T}{\partial\phi} + \dot{e}_{gen}$$

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Steady 1-D Sphere

$$\cancel{\rho c_p \frac{\partial T}{\partial t}} = \frac{1}{r^2}\frac{\partial}{\partial r}kr^2\frac{\partial T}{\partial r} + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}k\sin\theta\frac{\partial T}{\partial\theta}} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}k\frac{\partial T}{\partial\phi}} + \dot{e}_{gen}$$

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}kr^2\frac{\partial T}{\partial r} + \dot{e}_{gen}$$

For no heat generation

$$0 = \frac{d}{dr}kr^2\frac{dT}{dr}$$

$$kr^2\frac{dT}{dr} = C$$

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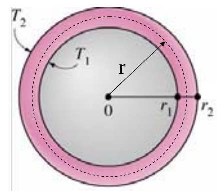
Spherical Shell



$$\dot{Q}_r = \dot{q}_r A = -k\frac{dT}{dr}4\pi r^2 = -4\pi C$$

$$\frac{\dot{Q}_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{\dot{Q}_r}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = -\int_{T_1}^{T_2}kdT$$

$$kr^2\frac{dT}{dr} = C$$

For constant k

$$\dot{Q}_r = \frac{4\pi k(T_1 - T_2)}{\frac{1}{r_1} - \frac{1}{r_2}} = \frac{T_1 - T_2}{\frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

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Steady, 1-D, Constant k, $\dot{e}_{gen} = 0$

- Rectangular

$$\dot{Q} = \frac{kA(T_1 - T_2)}{L} \Rightarrow \dot{Q} = \frac{T_1 - T_2}{R} \Rightarrow R = \frac{L}{kA}$$

- Cylindrical shell

$$\dot{Q} = \frac{2\pi kL(T_1 - T_2)}{\ln(r_2/r_1)} = \frac{T_1 - T_2}{R} \Rightarrow R = \frac{\ln(r_2/r_1)}{2\pi kL}$$

- Spherical shell

$$\dot{Q} = \frac{4\pi k(T_1 - T_2)}{1/r_1 - 1/r_2} = \frac{T_1 - T_2}{R} \Rightarrow R = \frac{1/r_1 - 1/r_2}{4\pi k}$$

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Steady, 1-D, Variable k, $\dot{e}_{gen} = 0$

- Rectangular $\dot{Q} = -\frac{A}{L}\int_{T_0}^{T_L}kdT \quad \dot{q} = \frac{\dot{Q}}{A} = -\frac{1}{L}\int_{T_0}^{T_L}kdT$

- Cylindrical shell

$$\dot{Q} = -\frac{2\pi L}{\ln(r_2/r_1)}\int_{T_1}^{T_2}kdT \quad \frac{\dot{Q}}{L} = -\frac{2\pi}{\ln(r_2/r_1)}\int_{T_1}^{T_2}kdT$$

- Spherical shell

$$\dot{Q} = -\frac{4\pi}{1/r_1 - 1/r_2}\int_{T_1}^{T_2}kdT$$

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Average Thermal Conductivity

- All the equations on the previous chart had an integral of thermal conductivity that is in the general form of an average
- If y = y(x) then y_{avg}, the average value of y between x_1 and x_2, is $y_{avg} = \bar{y} = \frac{1}{x_2 - x_1} = \int_{x_1}^{x_2} y dx$
- Applied to thermal conductivity, this general result is $k_{avg} = \bar{k} = \frac{1}{T_2 - T_1} = \int_{T_1}^{T_2} k dT$

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Steady, 1-D, Variable k, $\dot{e}_{gen} = 0$

- Rectangular $\dot{Q} = -\frac{A}{L}\int_{T_0}^{T_L} kdT \quad \dot{q} = \frac{\dot{Q}}{A} = -\frac{\bar{k}(T_L - T_0)}{L}$
- Cylindrical shell

$$\dot{Q} = -\frac{2\pi L}{\ln(r_2/r_1)}\int_{T_1}^{T_2} kdT \quad \frac{\dot{Q}}{L} = -\frac{2\pi\bar{k}(T_2 - T_1)}{\ln(r_2/r_1)}$$

- Spherical shell

$$\dot{Q} = -\frac{4\pi}{1/r_1 - 1/r_2}\int_{T_1}^{T_2} kdT = -\frac{4\pi\bar{k}(T_2 - T_1)}{1/r_1 - 1/r_2}$$

The formulas are the same as those for constant k if a suitable average is used

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1-D, Rectangular, Heat Generation

$$\rho c_p \cancel{\frac{\partial T}{\partial t}} = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} + \cancel{\frac{\partial}{\partial y} k \frac{\partial T}{\partial y}} + \cancel{\frac{\partial}{\partial z} k \frac{\partial T}{\partial z}} + \dot{e}_{gen}$$

0 for steady heat transfer

0 for one dimensional heat transfer

$$\frac{d}{dx} k \frac{dT}{dx} + \dot{e}_{gen} = 0 \quad \Rightarrow \quad -k\frac{dT}{dx} = \int \dot{e}_{gen} dx + C_1$$

$$-\int kdT = \int \left[\int \dot{e}_{gen} dx + C_1\right] dx + C_2$$

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1-D, Rectangular, Heat Generation

$$-\int kdT = \int \left[\int \dot{e}_{gen} dx + C_1\right] dx + C_2$$

- How do we find C_1 and C_2?
  - Have to match boundary conditions (at x = 0 and x = L) given in a particular problem
    - Can specify temperature at 0, L, or both
    - Can specify $\dot{q}$ = –kdT/dx at 0 or L, but not both
    - Can specify –kdT/dx = h(T - T_∞) at 0, L, or both
    - Can specify combinations of above conditions
  - Look at constant k and $\dot{e}_{gen}$ here

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1-D, Rectangular, Heat Generation

- For constant k and $\dot{e}_{gen}$ we can integrate the previous equation two times

$$-\int kdT = -kT = \int \left[\int \dot{e}_{gen} dx + C_1\right] dx + C_2 = \int \left[\dot{e}_{gen} x + C_1\right] dx + C_2$$

$$-kT = \int \left[\dot{e}_{gen} x + C_1\right] dx + C_2 = \frac{\dot{e}_{gen} x^2}{2} + C_1 x + C_2$$

- How do we get C_1 and C_2 if we know T = T_0 at x = 0 and T = T_L at x = L?

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1-D, Rectangular, Heat Generation

- For T = T_0 at x = 0 we must have

$$-kT_0 = \frac{\dot{e}_{gen} 0^2}{2} + C_1 0 + C_2 \quad \Rightarrow \quad -kT_0 = C_2$$

- For T = T_L at x = L we must have

$$-kT = \frac{\dot{e}_{gen} x^2}{2} + C_1 x + C_2 \quad \Rightarrow \quad -kT_L = \frac{\dot{e}_{gen} L^2}{2} + C_1 L + C_2$$

$$-kT_L = \frac{\dot{e}_{gen} L^2}{2} + C_1 L - kT_0 \quad \Rightarrow \quad C_1 = \frac{k(T_0 - T_L)}{L} - \frac{\dot{e}_{gen} L}{2}$$

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1-D, Rectangular, Heat Generation

- Substitute C_1 and C_2 into general solution

$$-kT = \frac{\dot{e}_{gen}x^2}{2} + C_1 x + C_2$$

$$-kT = \frac{\dot{e}_{gen}x^2}{2} + \left(\frac{k(T_0 - T_L)}{L} - \frac{\dot{e}_{gen}L}{2}\right)x - kT_0$$

$$\boxed{T = T_0 - \frac{\dot{e}_{gen}x^2}{2k} + \frac{\dot{e}_{gen}xL}{2k} - \frac{(T_0 - T_L)x}{L}}$$

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1-D, Rectangular, Heat Generation

- Write last equation in terms of dimensionless temperature ratio and x/L

$$\frac{T - T_0}{T_L - T_0} = \frac{x}{L} - \frac{\dot{e}_{gen}x(L - x)}{2k(T_L - T_0)} = \frac{x}{L} - \frac{L^2\dot{e}_{gen}}{2k(T_L - T_0)}\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

- Define dimensionless heat generation $H = \dfrac{L^2\dot{e}_{gen}}{k(T_L - T_0)}$

$$\frac{T - T_0}{T_L - T_0} = \frac{x}{L} - \frac{H}{2}\frac{x}{L}\left(1 - \frac{x}{L}\right) = \frac{x}{L}\left[1 - \frac{H}{2}\left(1 - \frac{x}{L}\right)\right]$$

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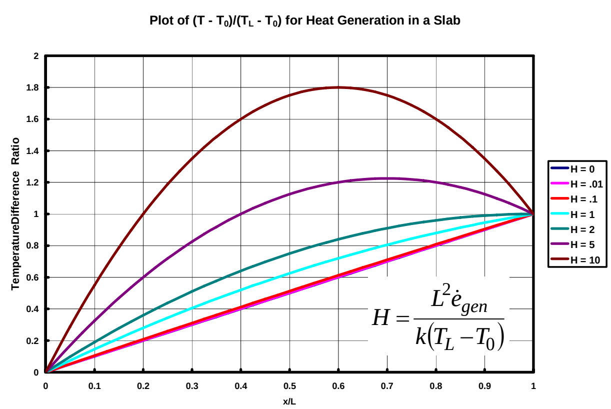

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1-D, Rectangular, Heat Generation

- Compute the heat flux from the boxed temperature equation on chart 34

$$\boxed{T = T_0 - \frac{\dot{e}_{gen}x^2}{2k} + \frac{\dot{e}_{gen}xL}{2k} - \frac{(T_0 - T_L)x}{L}}$$

$$\dot{q} = -k\frac{dT}{dx} = -k\left[-\frac{\dot{e}_{gen}2x}{2k} + \frac{\dot{e}_{gen}L}{2k} - \frac{(T_0 - T_L)}{L}\right]$$

$$\dot{q} = \frac{\dot{e}_{gen}(2x - L)}{2} + \frac{k(T_0 - T_L)}{L}$$

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Verify Heat Balance

- (Heat in at x = 0) + (Heat generated) = (Heat out at x = L) $\dot{Q}_{x=0} + \dot{E}_{gen} = \dot{Q}_{x=L}$
  - Look at a slab with thickness, L, and cross sectional area, A, giving a volume LA

$$\dot{q} = \frac{\dot{e}_{gen}(2x - L)}{2} + \frac{k(T_0 - T_L)}{L}$$

$$\dot{Q}_{x=0} = \dot{q}_{x=0}A = -\frac{\dot{e}_{gen}LA}{2} + \frac{kA(T_0 - T_L)}{L} \qquad \dot{E}_{gen} = LA\dot{e}_{gen}$$

$$\dot{Q}_{x=L} = \dot{q}_{x=L}A = \frac{\dot{e}_{gen}LA}{2} + \frac{kA(T_0 - T_L)}{L}$$

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What if $T_0 = T_L = T_B$

$$\boxed{T = T_0 - \frac{\dot{e}_{gen}x^2}{2k} + \frac{\dot{e}_{gen}xL}{2k} - \frac{(T_0 - T_L)x}{L}}$$

$$\dot{q} = \frac{\dot{e}_{gen}(2x - L)}{2} + \frac{k(T_0 - T_L)}{L}$$

- Setting $T_0 = T_L = T_B$ in general equations above gives

$$\boxed{T = T_B - \frac{\dot{e}_{gen}x^2}{2k} + \frac{\dot{e}_{gen}xL}{2k}} \qquad \dot{q} = \frac{\dot{e}_{gen}(2x - L)}{2}$$

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$T_0 = T_L = T_B$ Manipulations

- Setting $\dot{q} = 0$ and solving for x gives location of maximum temperature
  - Recall that $\dot{q} = -kdT/dx$ so $dT/dx = 0$ if $\dot{q} = 0$
  - Find that $x = L/2$ for maximum temperature

$$\frac{T_{max}}{T_B} = 1 + \frac{\dot{e}_{gen} L^2}{8kT_B}$$

- Dimensionless temperature results

$$\frac{T}{T_B} = 1 + \frac{\dot{e}_{gen} L^2}{2kT_B}\frac{x}{L}\left(1 - \frac{x}{L}\right) \quad and \quad \frac{T - T_B}{T_{max} - T_B} = 4\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

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See derivation slides at end of lecture

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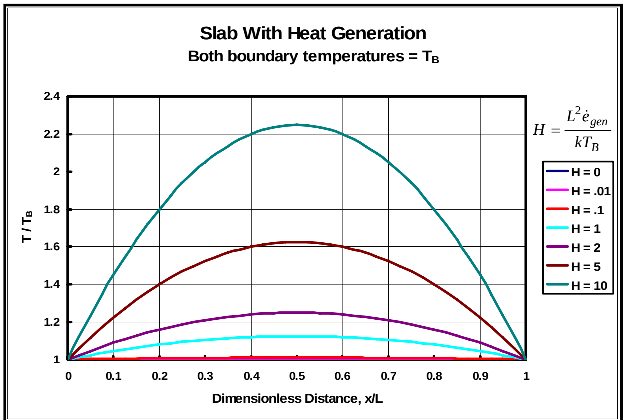

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Other Geometries

- Can find similar results with heat generation for solid cylinders and spheres, spherical shells and cylindrical shells
  - Same general approach, but different results for each type of geometry
  - See printed notes to get results for various geometries
    - Temperature, heat flow, maximum temperature, conditions for maximum temperature

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Solid Cylinder

- A solid cylinder with radius, R, constant heat generation and constant k, has a maximum temperature at its center

$$T_{max} - T_{surface} = \frac{\dot{e}_{gen} R^2}{4k}$$

- Chart 6 example had heat generation of 2.788×10^6 W/m³ for a 0.001 m diameter copper wire with a current of 10 A. What is $T_{max} - T_{surface}$ for this wire?

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Solution

- Take k = 403 W/m·K at 20°C

$$T_{max} - T_{surface} = \frac{\dot{e}_{gen} R^2}{4k} = \frac{\frac{2.788 x 10^6\, W}{m^3}(0.0005\, m)^2}{4\frac{403\, W}{m \cdot K}} = 0.0004K$$

- Can combine equations for T_{max} and $\dot{e}_{gen}$

$$T_{max} - T_{surface} = \frac{\dot{e}_{gen} R^2}{4k} \qquad \dot{e}_{gen} = \frac{I^2 \rho}{A^2} = \frac{I^2 \rho}{\pi^2 R^4}$$

$$T_{max} - T_{surface} = \frac{I^2 \rho}{4kR^2}$$

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Additional Charts

- The results shown in chart 37 are derived in the following slides
  - These charts show the algebraic details for the following results
    - Location of the maximum temperature
    - Value of the maximum temperature
    - Dimensionless forms of the temperature equation
- Additional version of the $T_{max} - T_{surface}$ equation is also presented

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Chart 37 Manipulation Details

- Start with basic result from chart 36

$$\boxed{T = T_B - \frac{\dot{e}_{gen}x^2}{2k} + \frac{\dot{e}_{gen}xL}{2k}}$$

- Divide by T_B and multiply terms on left by L/L or L^2/L^2 then rearrange to get

$$\frac{T}{T_B} = 1 - \frac{\dot{e}_{gen}x^2}{2kT_B}\frac{L^2}{L^2} + \frac{\dot{e}_{gen}xL}{2kT_B}\frac{L}{L} = 1 + \frac{\dot{e}_{gen}L^2}{2kT_B}\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

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Chart 37 Manipulation Details II

- Set dT/dx = 0 in chart 36 equation

$$-k\frac{dT}{dx} = \dot{q} = \frac{\dot{e}_{gen}(2x - L)}{2} = 0 \quad \Rightarrow \quad x_{T_{\max}} = \frac{L}{2}$$

- Substitute this x value into T equation to get the maximum temperature

$$\frac{T_{\max}}{T_B} = \left\{1 + \frac{\dot{e}_{gen}L^2}{2kT_B}\frac{x}{L}\left(1 - \frac{x}{L}\right)\right\}_{x=\frac{L}{2}} = 1 + \frac{\dot{e}_{gen}L^2}{8kT_B}$$

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Chart 37 Manipulation Details III

- Compare T/T_B and T_{max}/T_B equations

$$\frac{T}{T_B} = 1 + \frac{\dot{e}_{gen}L^2}{2kT_B}\frac{x}{L}\left(1 - \frac{x}{L}\right) \quad \Rightarrow \quad \frac{T}{T_B} - 1 = \frac{\dot{e}_{gen}L^2}{2kT_B}\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

$$\frac{T_{\max}}{T_B} = 1 + \frac{\dot{e}_{gen}L^2}{8kT_B} \quad \Rightarrow \quad \frac{\dot{e}_{gen}L^2}{2kT_B} = 4\left(\frac{T_{\max}}{T_B} - 1\right)$$

$$\frac{T}{T_B} - 1 = 4\left(\frac{T_{\max}}{T_B} - 1\right)\frac{x}{L}\left(1 - \frac{x}{L}\right) \quad \Rightarrow \quad \frac{T - T_B}{T_{\max} - T_B} = 4\frac{x}{L}\left(1 - \frac{x}{L}\right)$$

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More on $T_{max} - T_{surface}$

- Last equation on chart 41 $\quad T_{\max} - T_{surface} = \dfrac{I^2\rho}{4kR^2}$
- Weidemann-Franz Law (approximate) for metals: L = ρk/T = 2.45×10^{-8} ohm·W/K²
  - L is called the Lorentz constant
  - Experimental data agree to better than 10%

$$T_{\max} - T_{surface} = \frac{LI^2T}{4k^2R^2} = \frac{\left(\dfrac{2.45x10^{-8}\ ohm\cdot W}{K^2}\right)I^2T}{4k^2R^2}$$

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