Unit Nine – Entropy Calculations for Ideal Gases

Mechanical Engineering 370

Thermodynamics

Larry Caretto

November 2, 2010

Outline

- Quiz eight solution
- Review
- Goals for unit nine
- Calculating entropy with ideal gases
- Constant and variable heat capacity
- Isentropic calculations
- Use of equations and ideal gas tables
  - Air tables *versus* tables with less data

Review

- Entropy is a property
- The maximum work is done in a reversible process
  - With sign convention for work on an input device this is the minimum work input
- In an adiabatic reversible process the maximum work is done when $\Delta s = 0$
  - Find inlet/initial entropy at initial state
  - Final/outlet state determined by entropy and another final/outlet property

---

First Unit Nine Goal

- As a result of studying this unit you should be able to compute entropy changes in ideal gases for any change of state
  - using constant heat capacities
  - using equations that give heat capacities as a function of temperature
  - using ideal gas tables

Second Unit Nine Goal

- As a result of studying this unit you should be able to compute the end states of isentropic processes in ideal gases
  - using constant heat capacities
  - using equations that give heat capacities as a function of temperature
  - using ideal gas tables

Derive Ideal Gas Entropy

- Entropy defined as $ds = (du + Pdv)/T$
- Multiply by T: $\boxed{Tds = du + Pdv}$
- $Tds - \cancel{Pdv} = du = d(h - Pv) = dh - \cancel{Pdv} - vdP$ so $\boxed{Tds = dh - vdP}$
- Ideal gas: $Pv = RT$, $du = c_v dT$, $dh = c_p dT$
- For ideal gases $ds = c_v dT/T + R dv/v$
- For ideal gases $ds = c_p dT/T - R dP/P$
- Integrate last two equations to get Δs

Ideal Gas Entropy Change

- $dS = c_v dT/T + R dv/v = c_p dT/T - R dP/P$
- Integrate between two states (1 and 2)

$$s_2 - s_1 = \int_{T_1}^{T_2} c_v \frac{dT}{T} + R\ln\left(\frac{v_2}{v_1}\right)$$

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R\ln\left(\frac{P_2}{P_1}\right)$$

Handling $c_p(T) = c_v(T) + R$

- Easiest case is constant heat capacity

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

- If we have an equation for $c_p(T)$ or $c_v(T)$ we can integrate $c_p dT/T$ or $c_v dT/T$
  - We only need one equation since $c_p(T)$ and $c_v(T)$ are related by $c_p(T) = c_v(T) + R$
- Ideal gas tables give integral of $c_p dT/T$ called s^o as described next

Ideal Gas Entropy Tables

- Define $s^o(T) = \int_{T_0}^{T} c_p(T')\frac{dT'}{T'}$
- So that $s^o(T_2) - s^o(T_1) = \int_{T_1}^{T_2} c_p(T')\frac{dT'}{T'}$

$$= \int_{T_0}^{T_2} c_p(T')\frac{dT'}{T'} - \int_{T_0}^{T_1} c_p(T')\frac{dT'}{T'}$$

$$s_2 - s_1 = s^o(T_2) - s^o(T_1) - R\ln\left(\frac{P_2}{P_1}\right)$$

Example

- Air is heated from 300 K to 500 K at constant pressure. What is Δs?
- Compute result using air tables and repeat with constant heat capacity

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{v_2}{v_1}\right) = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right)$$

$$\Delta s = s^o(T_2) - s^o(T_1) - R\ln\left(\frac{P_2}{P_1}\right) = s^o(T_2) - s^o(T_1) + R\ln\left(\frac{v_2 T_1}{v_1 T_2}\right)$$

Example Answers

- Air is heated from 300 K to 500 K at constant pressure. What is Δs?
- From table A-17, page 936, s^o(300 K) = 1.70203 kJ/kg·K and s^o(500 K) = 2.21952 kJ/kg·K; Δs = 0.51749 kJ/kg·K
- Assuming constant c_p = 1.005 kJ/kg·K gives $\Delta s = c_p \ln(T_2/T_1)$ = 1.005 ln(500/300) = 0.51338 kJ/kg·K

Second Example

- Air is heated from 300 K, 100 KPa to 500 K, 200 KPa. What is Δs using the air tables?

$$\Delta s = s^o(500\ K) - s^o(300\ K) - \frac{0.287\ kJ}{kg\cdot K}\ln\left(\frac{200\ kPa}{100\ kPa}\right)$$

$$\Delta s = \frac{2.21952\ kJ}{kg\cdot K} - \frac{1.70203}{kg\cdot K} - \frac{0.287\ kJ}{kg\cdot K}\ln(2)$$

$$\Delta s = \frac{0.31856\ kJ}{kg\cdot K}$$

Third Example

- Air is heated from 300 K, 100 KPa to 500 K, 200 KPa. What is Δs assuming constant heat capacity?

$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) = \frac{1.005\ kJ}{kg\cdot K}\ln\left(\frac{500\ K}{300\ K}\right) - \frac{0.287\ kJ}{kg\cdot K}\ln\left(\frac{200\ kPa}{100\ kPa}\right)$$

$$\Delta s = \frac{0.51338\ kJ}{kg\cdot K} - \frac{.19893\ kJ}{kg\cdot K} = \frac{0.31444\ kJ}{kg\cdot K}$$

- Air table result $\Delta s = \frac{0.31856\ kJ}{kg\cdot K}$

Derive Isentropic Ideal Gas

- Start with ds = c_pdT/T – RdP/P
- For ds = 0, c_pdT/T = RdP/P
- Integrate for ds = 0 and constant c_p to get $c_p\ln(T_2/T_1) = R\ln(P_2/P_1)$ giving
- $\ln(T_2/T_1) = \ln(P_2/P_1)^{R/Cp}$ or $T_2/T_1 = (P_2/P_1)^{R/Cp}$
- $R/c_p = (c_p – c_v)/c_p = (c_p/c_v – 1)/(c_p/c_v)$
- $R/c_p = (k – 1)/k$, where $k = c_p/c_v$

Ideal Gas Isentropic Processes

- Final result: $T_2/T_1 = (P_2/P_1)^{(k-1)/k}$
  - $T/P^{(k-1)/k}$ = constant
- Can derive similar equations for other variables
  - $T_2/T_1 = (v_1/v_2)^{(k-1)}$ => Tv^{k-1} = constant
  - $P_2/P_1 = (v_1/v_2)^k$ => Pv^k = constant
- Apply only to ideal gases with constant heat capacities in isentropic processes

Variable Heat Capacity

- Set $s_2 – s_1 = 0$ in equations below for ideal gas isentropic process

$$s_2 - s_1 = \int_{T_1}^{T_2} c_v \frac{dT}{T} + R\ln\left(\frac{v_2}{v_1}\right) = 0$$

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R\ln\left(\frac{P_2}{P_1}\right) = 0$$

Variable Heat Capacity II

- What if we have an equation for $c_P(T)$ or $c_v(T) = c_P(T) – R$?

$$\int_{T_1}^{T_2} c_v \frac{dT}{T} = -R\ln\left(\frac{v_2}{v_1}\right) \qquad \int_{T_1}^{T_2} c_p \frac{dT}{T} = R\ln\left(\frac{P_2}{P_1}\right)$$

- Can solve for volume or pressure ratios if T_1 and T_2 are given
- A trial-and-error solution is required if T_1 or T_2 are unknown

s^o for Isentropic Processes

$$s_2 - s_1 = s^o(T_2) - s^o(T_1) - R\ln(P_2/P_1)$$

- $s_2 – s_1 = 0$ for isentropic process

$$s^o(T_2) = s^o(T_1) - R\ln(P_2/P_1)$$

- Given T_1 and P_2/P_1
  - find $s^o(T_1)$ in tables
  - Compute $s^o(T_2) = s^o(T_1) + R\ln(P_2/P_1)$
  - Interpolate in tables to find T_2 that gives s^o
- Need to solve by iteration if given v_2/v_1 instead of P_2/P_1
  - Use $s^o(T_2) = s^o(T_1) + R\ln(v_1T_2/v_2T_1)$

$P_r(T)$ for Isentropic Process

- For $s_2 = s_1$ $\quad s^o(T_2) = s^o(T_1) - R\ln(P_2/P_1)$
- Solve for P_2/P_1 and define $P_r(T) = A\exp[s^o(T)/R]$ $\quad \dfrac{s^o(T_2) - s^o(T_1)}{R} = \ln\dfrac{P_2}{P_1}$
  - A is an arbitrary constant that cancels $\quad \dfrac{P_2}{P_1} = e^{\frac{s^o(T_2)-s^o(T_1)}{R}} = \dfrac{e^{\frac{s^o(T_2)}{R}}}{e^{\frac{s^o(T_1)}{R}}} = \dfrac{P_r(T_2)}{P_r(T_1)}$
- We can tabulate the function $P_r(T)$ such that $P_r(T_2) = (P_2/P_1)P_r(T_1)$ relates the end states for $s_2 = s_1$

$v_r(T)$ for Isentropic Process

- We can tabulate the function $P_r(T)$ such that $P_r(T_2) = (P_2/P_1)P_r(T_1)$ relates the end states for $s_2 = s_1$
- If we know $v_2/v_1 = (T_2/T_1)(P_1/P_2)$ can use

$$\frac{P_r(T_2)}{P_r(T_1)} = \frac{P_2}{P_1} = \frac{T_2}{T_1}\frac{v_1}{v_2} \quad \Rightarrow \quad \frac{v_1}{v_2} = \frac{T_1}{T_2}\frac{P_r(T_2)}{P_r(T_1)} = \frac{\dfrac{T_1}{P_r(T_1)}}{\dfrac{T_2}{P_r(T_2)}} = \frac{v_r(T_1)}{v_r(T_2)}$$

- Tabulate function $v_r(T)$ such that $v_r(T_2) = (v_2/v_1)v_r(T_1)$ gives end states for $s_2 = s_1$

Variable Heat Capacity Tables

- For ideal gas, isentropic processes, with variable heat capacities we can define $P_r(T)$, and $v_r(T)$ such that
  - $v_2/v_1 = v_r(T_2)/ v_r(T_1)$
  - $P_2/P_1 = P_r(T_2)/ P_r(T_1)$

Relates end states such that $s(T_2,P_2) = s(T_1,P_1)$

- Values of $P_r(T)$, and $v_r(T)$ given in ideal gas tables
- We still use $Pv = RT$ at individual data points

Example Problem

- Adiabatic, steady-flow air compressor used to compress 10 kg/s of air from 300 K, 100 kPa to 1 MPa. What is the minimum work input?
- Minimum work input for adiabatic process is when process is isentropic
- Steady first law for one inlet, one outlet, negligible KE and PE:

$$\dot{Q} = \dot{W}_u + \dot{m}(h_{out} - h_{in})$$

Example Continued

- What is outlet state for maximum work? Use ideal gas tables for air on page 934
  - $P_r(300\text{ K}) = 1.3860$
  - $P_2/P_1 = P_r(T_2)/ P_r(T_1)$ so that
  - $P_r(T_2) = P_r(T_1)\, P_2/P_1 = 1.3860(1000/100)$
  - What is T with $P_r = 13.860$
  - $P_r(570\text{ K}) = 13.50$; $P_r(580\text{ K}) = 14.38$
  - Interpolate to get $P_r(T_2) = 13.86$ at $T_2 = 574.1$ K

Example Concluded

- Use enthalpy data from ideal gas tables
  - $h_{in} = h(300\text{ K}) = 300.19$ kJ/kg
  - $h_{out} = h(574.1\text{ K}) = 579.86$ kJ/kg (interpolated)

$$\dot{W}_u = \dot{m}(h_{in} - h_{out}) =$$

$$\frac{10\ kg}{s}\left(\frac{300.19\ kJ}{kg} - \frac{579.87\ kJ}{kg}\right)\frac{1\ MW\cdot s}{1000\ kJ} = -2.797\ MW$$

- Negative value shows work input
- Work input of 2.797 MW is minimum

Repeat with Constant c_p

- Here we use data for air at 300 K: k = 1.4, c_p = 1.005 kJ/kg·K `(page 911)`
- $T_2 = T_1(P_2/P_1)^{(k-1)/k}$ = (300 K)[(1000 kPa)/(100 kPa)]$^{(1.4-1)/1.4}$ = 579.2 K

$$\dot{W}_u = \dot{m}(h_{in} - h_{out}) = \dot{m}c_p(T_{in} - T_{out}) =$$

$$\frac{10\,kg}{s}\frac{1.005\,kJ}{kg \cdot K}(300\,K - 579.2\,K)\frac{1\,MW \cdot s}{1000\,kJ} = -2.806\,MW$$

Class Exercise

- Air in an adiabatic piston-cylinder device with $V_1 = 0.1\ m^3$, P_1 = 50 kPa, and T_1 = 300 K is compressed to $V_2 = 0.01\ m^3$. Find the minimum work input
- Do the calculation for variable heat capacity and repeat with constant c_p
  - – Variable: $v_2/v_1 = v_r(T_2)/ v_r(T_1)$ and w = –Δu
  - – Constant: $T_2/T_1 = (v_1/v_2)^{(k-1)}$ and w = – $c_v\Delta T$

Constant Heat Capacity Solution

- **State data:** $V_1 = 0.1\ m^3$, P_1 = 50 kPa, T_1 = 300 K, $V_2 = 0.01\ m^3$
- **First law:** Q = ΔU + W = $m(u_2 – u_1)$ + W so W = $m(u_1 – u_2)$ for Q = 0
- **Second law:** Maximum work in adiabatic process is when Δs = 0
- **Properties:** Air as ideal gas with constant heat capacity (c_p = 1.005 kJ/kg·K, c_v = 0.718 kJ/kg·K, R = 0.287 kJ/kg·K,k = 1.4 at 300K, p 911)

Constant Heat Capacity Solution II

- **Ideal gas equations:** PV = mRT, du = c_vdT, dh= c_pdT; for Δs = 0 with constant heat capacity, $T_2 = T_1(v_1/v_2)^{k-1}$
- **Solution:**

$$T_2 = T_1\left(\frac{v_2}{v_1}\right)^{k-1} = (300\,K)\left(\frac{0.1\,m^3}{0.01\,m^3}\right)^{1.4-1} = 753.6\,K$$

$$W = m(u_1 - u_2) = m\int_{T_2}^{T_1} c_v dT = mc_v(T_1 - T_2)$$

Constant Heat Capacity Solution III

- **Solution concluded:**

$$m = \frac{P_1V_1}{RT_1} = \frac{(50\,kPa)(0.1\,m^3)}{\frac{0.287\,kPa \cdot m^3}{kg \cdot K}(300\,K)} = 0.05807\ \ kg$$

$$W = m(u_1 - u_2) = m\int_{T_2}^{T_1} c_v dT = mc_v(T_1 - T_2)$$

$$= (0.05807\,kg)\frac{0.718\,kJ}{kg \cdot K} = (300\,K - 753.6\,K) = -18.9\,kJ$$

Minimum work input is 18.9 kJ

Variable Heat Capacity Solution

- **State data:** $V_1 = 0.1\ m^3$, P_1 = 50 kPa, T_1 = 300 K, $V_2 = 0.01\ m^3$
- **First law:** Q = ΔU + W = $m(u_2 – u_1)$ + W so W = $m(u_1 – u_2)$ for Q = 0
- **Second law:** Maximum work in adiabatic process is when Δs = 0
- **Properties:** Use ideal gas tables for air on page 936

Variable Heat Capacity Solution II

- **Ideal gas equations:**
  - PV = mRT,
  - $\Delta u = u(T_2) - u(T_1)$
  - $\Delta h = h(T_2) - h(T_1)$
  - For $\Delta s = 0$, $v_r(T_2) = v_r(T_1)(v_2/v_1) = v_r(T_1)(V_2/V_1)$

$$W = m(u_1 - u_2) = m[u(T_1) - u(T_2)]$$

- **Solution:** At T_1 = 300 K v_r = 621.2
  - $v_r(T_2) = v_r(T_1)(V_1/V_2)$ = 621.2(0.01 m^3) / (0.1 m^3) = 62.12
  - In tables v_r(730 K) = 62.13 so T_2 = 730 K

Variable Heat Capacity Solution III

- **Solution concluded:**

$$m = \frac{P_1V_1}{RT_1} = \frac{(50\ kPa)(0.1\, m^3)}{\frac{0.287\ kPa \cdot m^3}{kg \cdot K}(300\ K)} = 0.05807\ \ kg$$

- Air tables: $u(T_1$ = 300 K) = 214.07 kJ/kg·K and $u(T_2$ = 730 K) = 536.07 kJ/kg·K and

$$W = m[u(T_1) - u(T_2)] = (0.05807\ kg)\left(\frac{214.07\ kJ}{kg} - \frac{536.07\ kJ}{kg}\right) = -18.7\ kJ$$

Minimum work input is 18.7 kJ