California State University Northridge

College of Engineering and Computer Science
Mechanical Engineering Department

Mechanical Engineering 309
Numerical Analysis of engineering Systems

Spring 2014   Number: 15237   Instructor: Larry Caretto

Solution to Midterm Exam

1. (20 points) **The function mystery, listed below, is called from cell A2 of the worksheet shown at the left. (a) What value will cell A2 have after the function executes? (b) What value will cell B2 have if the function call in cell A2 is copied to cell B2?**

	A	B	C
1	2	5	
2	=mystery(B1,A1)		

```
Function mystery(x As Integer, _
           y As Double) As Double
    Dim c As Integer
    Dim d As Double
    Dim p As Double

    c = x
    d = y
    p = 1
    Do While c > 0
        If c Mod 2 <> 0 Then
            p = p * d
            c = c - 1
        Else
            d = d * d
            c = c / 2
        End If
    Loop
    mystery = p

End Function
```

For the current formula location, the formula becomes = mystery(5, 2) and we can trace the loop as follows:

The variables are initialized as follows: c = x = 5, d = y = 2, and p = 1. We now need to trace the action in the while loop.

c>0 is 5>0 which is true so we start the loop.
c Mod 2 =5 Mod 2 = 1 (≠ 0) so we do the top block of the if statement
p = p * d = 1 * 2 = 2
c = c – 1 = 5 – 1 = 4

The loop ends with c = 4 which is > 0 so we loop again
c Mod 2 = 4 Mod 2 = 0 so we do the else block
d = d * d = 2 * 2 = 4
c = c / 2 = 4 / 2 = 2

The loop ends with c = 2 which is > 0 so we loop again
c Mod 2 =2 Mod 2= 0 so we do the else block
d = d * d = 4 * 4 = 16
c = c / 2 = 2 / 2 = 1

The loop ends with c = 1 which is > 0 so we loop again
c Mod 2 =1 Mod 2= 1 so we do the top block
p = p * d = 2 * 16 = 32
c = 1 – 1 = 0

The loop ends with c = 0 so we exit the loop
Cell A2 contains the value of **mystery = p = 32**

When we copy the formula to cell B2, the formula becomes = mystery(C1, B1)..Since C1 is empty this becomes mystery(0,5).

In the program we initialize the variables to the following values: c = x= 0, d = y = 5, and p = 1.

Because c = 0 we never enter the while loop and return the value **mystery = p = 1** to cell B2.

2. (15 points) Answer the following questions. Give your reasoning where appropriate

a. (3 points) In a numerical calculation the step size is decreased by a factor of 1/3 and the error is decreased by a factor of 1/81. What is the order of the error for this operation?

Here a decrease by a factor of 3 decreases the error by a factor of 81 = 3^4. Thus the have a fourth order error.

b. (5 points) For the arrays A = [1 2 3] and B = [-1 0 1]^T, write the products AB and BA in the space below. If either of these products is not possible, explain why it not possible.

$$\boldsymbol{AB} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = [1(-1)+2(0)+3(1)] = [2] = 2$$

$$\boldsymbol{BA} = \begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}-1(1) & -1(2) & -1(3)\\0(1) & 0(2) & 0(3)\\1(1) & 1(2) & 1(3)\end{bmatrix} = \begin{bmatrix}-1 & -2 & -3\\0 & 0 & 0\\1 & 2 & 3\end{bmatrix}$$

c. (3 points) Is following statement is true or false? The secant method for solving f(x) = 0 requires two initial guesses that bracket the root. __FALSE__________. Give the reasoning for your answer. The secant method requires two initial guesses, but they do not have to bracket the root.

d. (4 points) The for loop below sets some values of the x array. Give the array index and the value of the array component [i.e. fill in the blanks in statements like x(_____) = ______] for each array component set in the for loop.

```
For k = 1 to 11 Step 3
    x(3*k+1) = (2*k)^2
Next k
```

For this loop we will have k = 1, 4, 7, and 10, giving x(4) = 4, x(13) = 64, x(22) = 196 and x(31) = 400.

3. (25 points) For each system of equations shown below determine if there is (i) a unique solution, (ii) an infinite number of solutions or (iii) no solution. If the solution is unique, find the unique solution. If there are an infinite number of solutions, find a general expression for these solutions; i.e. find the constants a, b, c, and d in the equations x = a + bz and y = c + dz.

(a)	(b)	(c)
x + 3y – 2z = –1	**x + 3y – 2z = –1**	**x + 3y – 2z = –1**
5x – 2y + z = 0	**5x – 2y + z = 0**	**5x – 2y + z = 0**
7x – 13y + 8z = 3	**7x – 13y + 8z = 4**	**7x – 13y = –13**

We can use Gaussian elimination to solve each set of equations. Set (a) we have:

```
1   3  -2  -1  Row 1      1   3  -2  -1  Row 1
5  -2   1   0  Row 2      0 -17  11   5  Row 2 – 5(Row 1)
7 -13   8   3  Row 3      0 -34  22  10  Row 3 – 7(Row 1)

1   3  -2  -1  Row 1
0 -17  11   5  Row 2
0   0   0   0  Row 3 – (-34/-17)(Row 2)
```

This solution has Rank(**A**) = Rank([**A b**]) = 2, which is less than the number of unknowns so we have an infinite number of solutions. If we set z = α, which can be any number, we find from the final second row that y = (5 – 11α)/(-17) and from the final first row, x = (-1 +2α -3(5 – 11α)/(-17)]/1 = (17 - 34 α - 15 + 33α)/(-17) = (2 – α)/(-17). Check by substituting these values into the third equation:7x – 13y + 8z = 7(2 – α)/(-17) – 13 (5 – 11α)/(-17) + 8α =(1/(-17)) (14 – 7 α – 65 + 143α – 136α) = –51/(-17) = 3, which is the correct right-hand side for the third equation.

For set (b) we have

```
1   3  -2  -1  Row 1      1   3  -2  -1  Row 1
```

```
5  -2   1   0  Row 2      0 -17  11   5  Row 2 – 5(Row 1)
7 -13   8   4  Row 3      0 -34  22  11  Row 3 – 7(Row 1)

1   3  -2  -1  Row 1
0 -17  11   5  Row 2
0   0   0   1  Row 3 – (-34/-17)(Row 2)
```

This solution has Rank(**A**) = 2 ≠ Rank([**A b**]) = 3, so there is no solution.

For set (c) we have

```
1   3  -2  -1  Row 1      1   3  -2  -1  Row 1
5  -2   1   0  Row 2      0 -17  11   5  Row 2 – 5(Row 1)
7 -13   0 -13  Row 3      0 -34  14  -6  Row 3 – 7(Row 1)

1   3  -2  -1  Row 1
0 -17  11   5  Row 2
0   0  -8 -16  Row 3 – (-34/-17)(Row 2)
```

This solution has Rank(**A**) = Rank([**A b**]) = 3, so the solution is unique. Back substitution gives z = (-16)/(-8) = **z = 2**, y = (5 – 11z)/(-17) = [5 – 11(2)] /(-17) = **y = 1** and x = [-1 –(-2)z – 3y]/1 = [-1 – (-2)(2) – 3(1)]/1 = **x = 0**

4. (20 points) The heat capacity of air as a function of temperature is shown in the table below.

C_p (kJ/kg·K)	1.005	1.013	1.029	1.051	1.075	1.099	1.121	1.142	1.155	1.173
T (K)	300	400	500	600	700	800	900	1000	1100	1200

Find the interpolated value of heat capacity at T = 750 K using a cubic polynomial: $a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$.

For a cubic polynomial we require four data points. The four points closest to T = 750 are T = 600, 700, 800, and 900. The divided difference table for these points is shown at the right. The calculations are shown below.

T	Cp	F	S	T
600	1.051			
		0.00024		
700	1.075		0	
		0.00024		-3.3×10^{-10}
800	1.099		-1e-7	
		0.00022		
900	1.121			

$$F_0 = \frac{1.075 - 1.051}{700 - 600} = 0.00024 \quad F_1 = \frac{1.099 - 1.075}{800 - 700} = 0.00024$$

$$F_2 = \frac{1.121 - 1.099}{900 - 800} = 0.00022 \quad S_0 = \frac{.00024 - .00024}{800 - 600} = 0$$

$$S_1 = \frac{.00022 - .00024}{900 - 700} = -10^{-7} \quad T_0 = \frac{-10^{-7} - 0}{900 - 600} = 3.3310^{-10}$$

We have $a_0 = y_0 = 1.051$, $a_1 = F_0 = 0.00024$, $a_2 = S_0 = 0$, and $a_3 = T_0 = 3.3\times10^{-10}$

Plugging these values into our Newton polynomial gives $a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$=1.051 + 0.00024(T – 600) + (0)(T – 600)(T – 700) + (-3.3×10^{-10})(T – 600)(T – 700)(T – 600)(T – 800) = 1.063

5. (25 points) Müller's method for solving equations uses a quadratic interpolation (or extrapolation) to find the next estimate of the root. It uses three data points, x_0, x_1, and x_2, at each iteration. It proceeds in the following manner.

1. **Start with three initial guesses, x_0, x_1, and x_2, and the computed values $f(x_0)$, $f(x_1)$, and $f(x_2)$.**
2. **Compute $\Delta x_0 = x_1 - x_0$ and $\Delta x_1 = x_2 - x_1$.**
3. **Compute** $d_0 = \dfrac{f(x_1) - f(x_0)}{\Delta x_0}$ **and** $d_1 = \dfrac{f(x_2) - f(x_1)}{\Delta x_1}$

4. Compute $a = \dfrac{d_1 - d_0}{\Delta x_0 + \Delta x_1}$, b = a$\Delta x_1$ + d_1, and c = f(x_2).

5. Compute $x_3 = x_2 - \dfrac{2c}{b \pm \sqrt{b^2 - 4ac}}$, where the + or – part of the ± sign is chosen to have the same sign as b; if b is negative pick the – sign; if b is positive, pick the + sign.

6. If x_3 is close enough to x_2, the solution is converged; the answer is x_3.

7. If x_3 is *not* close enough to x_2, compute f(x_3), update the guesses and function evaluations – replace x_0 by x_1, x_1 by x_2, x_2 by x_3, f(x_0) by f(x_1), f(x_1) by f(x_2), and f(x_2) by f(x_3) –and return to step two to continue the iterations.

Use Müller's method to find the smallest positive value of x where sin(x) = cos(x). Find the relative error in x after two iterations. Start with x = 0, 0.5, and 1 as your initial guesses.

<u>Choose f(x) = sin(x) – cos(x) as the equation to solve for sin(x) = cos(x). For the first iteration:</u>

1. x_0, = 0, x_1 = 0.5 and x_2 = 1 and the computed values f(x_0) = sin(0) – cos(0) = 1; f(x_1) sin(0.5) – cos(0.5) = -0.39816, and f(x_2) = sin(1) – cos(1) = 0.30117
2. $\Delta x_0 = x_1 - x_0$ =0.5 – 0 = 0.5 and $\Delta x_1 = x_2 - x_1$ = 1 – 0.5 = 0.5.
3. $d_0 = \dfrac{f(x_1) - f(x_0)}{\Delta x_0} = \dfrac{-0.39816 - (-1)}{0.5} = 1.2037$ and

   $d_1 = \dfrac{f(x_2) - f(x_1)}{\Delta x_1} = \dfrac{.301169 - (-.398160)}{0.5} = 1.38965$
4. $a = \dfrac{d_1 - d_0}{\Delta x_0 + \Delta x_1} = \dfrac{1.38965 - 1.2037}{0.5 + 0.5} = 0.194965$, b = a$\Delta x_1$ + d_1 = 0.194965(0.5) +1.38965, and c = f(x_2) = 0.30117
5. Since b is positive we use the + sign in front of the square root.

   $$x_3 = 1 - \frac{2(0.30117)}{1.38965 + \sqrt{1.38965^2 - 4(0.194965)(0.30117)}} = 0.793125$$
6. If x_3 is close enough to x_2, the solution is converged; the answer is x_3. Here we are told to take two iterations.
7. Update the guesses and function evaluations – use x_1 to update x_0 so that x_0 = 0.5, use x_2 to update x_1 so that x_1 =1, and use x_3, to update x_2 so that x_2 = 0.793125. With these values we find f(x_0) = -0.39816, the old f(x_1); f(x_1) = 0.30117, the old f(x_2), and f(x_2) = f(0.793215) = sin(0.793215) – cos(0.793215) = 0.010927.

<u>We now start the second iteration at step two to continue the iterations.</u>

2. $\Delta x_0 = x_1 - x_0$ = 1 – 0.5 = 0.5 and $\Delta x_1 = x_2 - x_1$ = 0.793215 – 1 = -0.20687.
3. $d_0 = \dfrac{f(x_1) - f(x_0)}{\Delta x_0} = \dfrac{0.30117 - (-0.39816)}{0.5} = 1.39865$ and

   $d_1 = \dfrac{f(x_2) - f(x_1)}{\Delta x_1} = \dfrac{0.010927 - (0.30117)}{-0.20687} = 1.40298$

4. $a = \frac{d_1 - d_0}{\Delta x_0 + \Delta x_1} = \frac{1.40298 - 1.39865}{0.5 - 0.20687} = 0.014765$, b = a$\Delta x_1$ + d_1 = 0.014765(-0.20687) +1.40298 = 1.399925, and c = f(x_2) = 0.010927

5. Since b is positive we use the + sign in front of the square root.

$$x_3 = 0.793125 - \frac{2(0.010927)}{1.39925 + \sqrt{1.399265^2 - 4(0.014765)(0.010927)}} = 0.785318$$

At this point we can find the relative error as the two successive values of x_3 divided by the most recent value. This gives the relative error as follows:

$$relErr = \frac{|\Delta x_3|}{|x_3|} = \frac{|0.75813 - 0.793125|}{|0.75813|} = 0.0099$$