Numerical Solutions of Ordinary Differential Equations

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Numerical Analysis of Engineering Systems

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Outline

- What is a differential equation
- Basic algorithms for numerical solution of differential equations
  - Euler, modified Euler, Huen, and Runge-Kutta
- Local *vs*. global truncation error
- Comparison of errors for different algorithms
- Advice: use higher-order algorithms for more accuracy with a given computer time

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What is a Differential Equation

- Look at examples you may have seen

Equations for velocity and acceleration $\frac{ds}{dt} = v \qquad \frac{d^2s}{dt^2} = \frac{F}{m}$

Electrical circuits $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV(t)}{dt}$

- Order of the equation is the order of the highest derivative in the equation
  - Equation for ds/dt is first order, other two are second order

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Ordinary Differential Equation

- Ordinary differential equations (ODE) have one independent variable
  - Partial differential equations (PDE) have more than one independent variable
  - In equations below t is the independent variable; dependent variables are s or I
  - Linear ODE: the dependent variable is in linear terms only (no s^2, sds/dt, *etc.*)

$$\frac{ds}{dt} = v \qquad \frac{d^2s}{dt^2} = \frac{F}{m} \qquad L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV(t)}{dt}$$

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Integration

- Evaluating an integral is the same as solving the equation dy/dx = f(x)
  - The answer is y = ∫f(x)dx + C, where C is found from some condition on $y_0(x_0)$
  - Solving an ODE is sometimes called integration of an ODE, but the problem is harder than solving a definite integral
  - The ODE has dy/dx = f(x,y) and we cannot solve y = ∫f(x)dx + C

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Solving ODEs

- Analytical techniques covered in courses on differential equations
- Numerical techniques used when conventional solutions are not possible
- Look at initial value problem (IVP)
  - dy/dx = f(x,y) with equation given for f(x,y) and known initial condition: $y(x_0) = y_0$
    - x_0 and y_0 given; very often $x_0 = 0$
    - Get solution for this problem
    - Extend to higher-order equations and systems of differential equations of any order

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The Initial Value Problem

- $dy/dx = f(x,y)$ with $y(x_0) = y_0$
- Basic numerical approach
  - Use a finite difference grid: $x_{i+1} – x_i = h_i$
  - Replace derivative by finite-difference approximation: $dy/dx \approx (y_{i+1} – y_i) / (x_{i+1} – x_i) = (y_{i+1} – y_i) / h_i$ (h may be a constant)
  - Derive a formula to compute f_{avg} the average value of f(x,y) between x_i and x_{i+1}
  - Replace $dy/dx = f(x,y)$ by $(y_{i+1} – y_i) / h_i = f_{avg}$
  - Repeatedly compute $y_{i+1} = y_i + h_i f_{avg}$

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Notation

- x_i is the value of the independent variable at point i on the grid
  - Determined from the user-selected value of step size $x_{i+1} = x_i + h$ (or h_i for variable step)
  - Can always specify exactly the independent variable's value, x_i
- y_i is the value of the numerical solution at the point where $x = x_i$
- f_i is derivative value found from x_i and the numerical value, y_i. I.e., $f_i = f(x_i, y_i)$.

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More Notation

- $y(x_i)$ is the exact value of y at $x = x_i$
  - Usually not known but notation is used in error analysis of algorithms
- $f(x_i,y(x_i))$ is the exact value of the derivative at $x = x_i$ (also not known)
- $e_1 = y(x_1) – y_1$ is the local truncation error for the first step of the algorithm
- $e_k = y(x_k) – y_k$ is the global truncation error after the k^{th} step of the algorithm

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Local *versus* Global Error

- At the initial condition we know the solution, y, exactly
- First step introduces some error
- Remaining steps have single step error plus previous accumulated error
- $E_i = y(x_i) – y_i$ is global truncation error at step i
  - Difference between numerical and exact solution after several steps
  - This is the error we want to control

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Euler's Method

- Simplest algorithm, example used for error analysis, not for practical use
- Define $f_{avg} = f(x_i, y_i)$
- Euler's method algorithm is $y_{i+1} = y_i + hf_i = y_i + hf(x_i, y_i)$
- Example $dy/dx = x + y$, $y = 0$ at $x = 0$
- Choose $h = 0.1$. What is y_1?
- We have $x_0 = 0$, $y_0 = 0$, $f_0 = x_0 + y_0 = 0$
  $x_1 = x_0 + h = 0.1$, $y_1 = y_0 + hf_0 = 0$

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Euler Example Continued

- Next step is from $x_1 = 0.1$ to $x_2 = 0.2$
- $f_1 = x_1 + y_1 = 0.1 + 0 = 0.1$ ($x_2 = x_1 + h$)
- $y_2 = y_1 + hf_1 = 0 + (.1)(.1) = .01$
- Can continue in this fashion
- For $dy/dx = x + y$, we know the exact solution: $y(x) = (x_0 + y_0 + 1)e^{x-x_0} – x – 1$
- For $x_0 = y_0 = 0$, $y(x) = e^x – x – 1$
- Look at application of Euler algorithm for a few steps and compute the error

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Euler Example

Previous results

x_i	y_i	f_i	$f(x_i,y(x_i))$	$y(x_i)$	$E(x_i)$
0	0	0	0	0	0
.1	0	.1	.1052	.0052	.0052
.2	.01	.21	.2214	.0214	.0114
.3	.031	.331	.3499	.04986	.01886
.4	.0641	.4641	.4918	.091825	.027725

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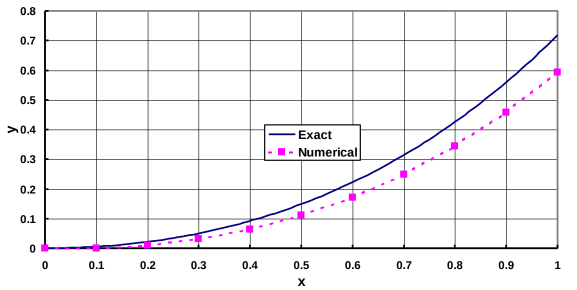
Euler Example Plot



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Error Propagation

- Behavior of Euler algorithm is typical of all algorithms for numerical solutions
- Error grows at each step
  - Error after several steps called global error
- We usually do not know this global error, but we would like to control it
- Look at local error for Euler algorithm
- Then discuss general relationship between local and global error

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Taylor Series to Get Error

- Expand y(x) in Taylor series about x = a

$$y(x) = y(a) + \left.\frac{dy}{dx}\right|_{x=a}(x-a) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=a}(x-a)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=a}(x-a)^3 + ...$$

- Look at first step from $x = x_0 = a$ to $x = x_0 + h = a + h$, so that $x - a = h$

$$y(x_0+h) = y(x_0) + \left.\frac{dy}{dx}\right|_0 h + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_0 h^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_0 h^3 + ...$$

- In ODE notation, $dy/dx|_0 = f(x_0, y(x_0))$

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Local Euler Error

- Result of Taylor series on last chart

$$y(x_0+h) = y(x_0) + hf(x_0, y(x_0)) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_0 h^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_0 h^3 + ...$$

Euler Algorithm 1st Step — Truncation Error

- This is only the Euler algorithm for the first step when we know $f(x_0,y(x_0))$
- This gives the local truncation error
- Local truncation error for Euler algorithm is second order, $O(h^2)$

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Global Error

- We will show that a local error of order n, has a global error of order n-1
- To show this consider the global error at $x = x_0 + kh$ after k algorithm steps
  - Is approximately k times the local error
  - If local error is $O(h^n)$, approximate global error after k steps is $k\,O(h^n) \approx kAh^n$
  - A new step size, h/r, takes kr steps to get to the same x value

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Global Error Concluded

- Compare error for same x = kh with step sizes h and h/r
- $E_{x=kh}(h) \approx kAh^n$
- $E_{x=kh}(h/r) \approx krA(h/r)^n$

$$\frac{E_{x=kh}\left(h/r\right)}{E_{x=kh}(h)} \approx \frac{kr\left(h/r\right)^n}{k(h)^n} = \frac{1}{r^{n-1}}$$

- When we reduce the step size by a factor of 1/r we reduce the error by a factor of $1/r^{n-1}$; this is the behavior of an algorithm whose error is order n-1

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Euler Local and Global Error

- Previously showed Euler algorithm to have second order local error
- Should have first order global error
- Results for previous Euler example at x = 1 with different step sizes

Step size	First step	Final error
h = 0.1	5.17×10^{-3}	1.25×10^{-1}
h = 0.01	5.02×10^{-5}	1.35×10^{-2}
h = 0.001	5.00×10^{-7}	1.36×10^{-3}

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Better Algorithms

- Seek high accuracy with low computational work
- Could improve Euler accuracy by cutting step size, but this is not efficient
- Use other algorithms that have higher order errors
- Runge-Kutta methods typically used
  - This is a class of methods that use several function evaluation methods per step

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Second-order Runge Kutta

- Huen's method

$$y_{i+1}^0 = y_i + h_i f(x_i, y_i) \qquad x_{i+1} = x_i + h_i$$

$$y_{i+1} = y_i + \frac{h_i}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right] = \frac{y_i + y_{i+1}^0 + h_i f(x_{i+1}, y_{i+1}^0)}{2}$$

- Modified Euler method

$$y_{i+1/2} = y_i + \left[\frac{h_i}{2}\right] f(x_i, y_i) \qquad x_{i+1/2} = x_i + \frac{h_i}{2}$$

$$y_{i+1} = y_i + h_i f(x_{i+1/2}, y_{i+1/2})$$

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Fourth-order Runge Kutta

- Uses four derivative evaluations per step

$$y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \qquad x_{i+1} = x_i + h_i$$

$$k_1 = h_i f(x_i, y_i)$$

$$k_2 = h_i f\left(x_i + \frac{h_i}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = h_i f\left(x_i + \frac{h_i}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = h_i f(x_i + h_i, y_i + k_3)$$

Values of k_1, k_2, k_3, and k_4, are estimates of $\Delta y = y_{i+1} - y_i$ based on derivative evaluations at x_i, $x_i+h_i/2$, $x_i+h_i/2$, and x_i+h_i

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Comparison of Methods

- Look at Euler, Heun, Modified Euler and fourth-order Runge-Kutta
- Solve $dy/dx = e^{-y-x}$ with $y(0) = 1$
- Compare numerical values to exact solution $y = \ln(e^{y_0} + e^{-x_0} - e^{-x})$
- Look at errors in the methods at x = 1 as a function of step size
- Plot error propagation (increase in error as x increases) for various methods

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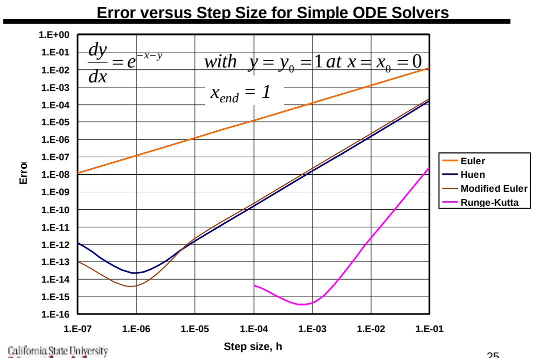

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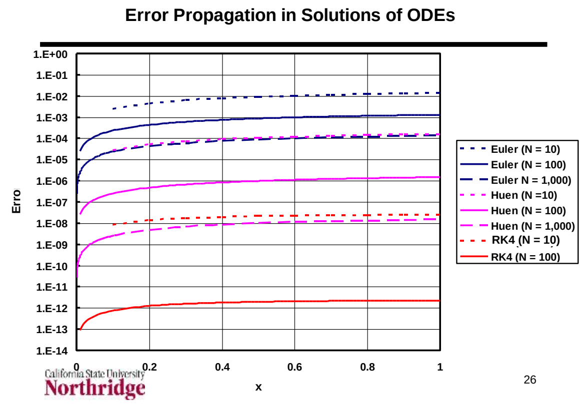

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Exercise Problem

- Apply modified Euler method (equations below) to dy/dx = x + y with y(0) = 1
  - Find solution at x = 0.6 using h = 0.2

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i) \qquad x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$
$$y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- For this problem dy/dx = f(x,y) = x + y
- We always have $x_{i+1} = x_i + h$
  - No need for h subscript for constant h

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Exercise Problem II

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i) \qquad x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$
$$y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- Given initial point: $y_0 = 1$ at $x_0 = 0$
  - $f = x + y$ so $f_0 = x_0 + y_0 = 0 + 1 = 1$
  - $y_{0+1/2} = y_0 + (h/2)f_0 = 1 + (0.2/2)(1) = 1.1$
  - $x_{0+1/2} = x_0 + (h/2) = 0 + (0.2/2) = 0.1$
  - $f(x_{0+1/2},y_{0+1/2}) = x_{0+1/2} + y_{0+1/2} = 0.1 + 1.1 = 1.2$
  - $y_1 = y_0 + hf(x_{0+1/2},y_{0+1/2}) = 1 + 0.2(1.2) = 1.24$
  - $x_1 = x_0 + h = 0 + 0.2 = 0.2$

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Exercise Problem III

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i) \qquad x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$
$$y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- After one step, $y_1 = 1.24$ at $x_1 = 0.2$
  - $f = x + y$ so $f_1 = x_1 + y_1 = 0.2 + 1.24 = 1.44$
  - $y_{1+1/2} = y_1 + (h/2)f_1 = 1.24 + (0.2/2)(1.44) = 1.384$
  - $x_{1+1/2} = x_1 + (h/2) = 0.2 + (0.2/2) = 0.3$
  - $f(x_{1+1/2},y_{1+1/2}) = x_{1+1/2} + y_{1+1/2} = 0.3 + 1.384 = 1.684$
  - $y_2 = y_1 + hf(x_{1+1/2},y_{1+1/2}) = 1.24 + 0.2(1.684) = 1.5768$
  - $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

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Exercise Problem IV

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i) \qquad x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$$
$$y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- After two steps, $y_2 = 1.5768$ at $x_2 = 0.4$
  - $f = x + y$ so $f_2 = x_2 + y_2 = 0.4 + 1.577 = 1.977$
  - $y_{2+1/2} = y_2 + (h/2)f_2 = 1.577 + (0.2/2)(1.977) = 1.774$
  - $x_{2+1/2} = x_2 + (h/2) = 0.4 + (0.2/2) = 0.5$
  - $f(x_{2+1/2},y_{2+1/2}) = x_{2+1/2} + y_{2+1/2} = 0.5 + 1.774 = 2.274$
  - $y_3 = y_2 + hf(x_{1+1/2},y_{1+1/2}) = 1.577 + 0.2(2.274) = 2.032$
  - $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$ (final step)

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