[redacted]

Name: [redacted]

Math 490, Spring 2009 Exam # 2

1. (5 pts) Is $\dfrac{\sqrt{\sqrt{\sqrt{7}+18}}}{\sqrt{3}-6}$ a constructible number? Explain your answer.

18, 7, 3, and -6 are constructible

$\Rightarrow$ $\sqrt{7}+18$ is constructible since constructable numbers are close under addition and square root of a
$\sqrt{3}-6$ +ve number is constructible

$\Rightarrow$ $\sqrt{\sqrt{7}+18}$ is constructible b/c sqrt is constructible

$\Rightarrow$ $\sqrt{\sqrt{\sqrt{7}+18}}$

$\Rightarrow$ $\dfrac{\sqrt{\sqrt{\sqrt{7}+18}}}{\sqrt{3}-6}$ is constructibe because the quotient of two constructible numbers is constructible

2. Determine whether each of the following is true or false. If true, prove it. Otherwise, give a counterexample.

a. (5 pts) Every algebraic number is a rational number.

False, $\sqrt{2}$ is algebraic (is a root of x^2-2) but $\sqrt{2}$ is not rational.

b. (5 pts) Every algebraic number is constructible (using straightedge and compass only).

$\sqrt[3]{2}$ is algebraic (a root of x^3-2) but $\sqrt[3]{2}$ is not constructible

c. (5 pts) Every rational number is an algebraic number.

Let $\frac{a}{b} \in \mathbb{Q}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$

Every integer is constructible because it is a multiple of 1. $\Rightarrow$ a, b are constructible.
Since constructible numbers is closed under division, $\frac{a}{b}$ is constructible. $\therefore$ Every rational number is con[illegible]
alg. (a)

20

3. (8 pts) Use the fact that $\cos(2\pi/9)$ is a root of the polynomial $f(x) = (x-1)(2x+1)(8x^3 - 6x + 1)$ to prove that a regular nine-sided polygon (a.k.a. a nonagon or regular 9-gon) is not constructible with straightedge and compass only.

We know that $\cos(2\pi/9) \neq 1$, so we can ignore $(x-1)$ since $\cos(2\pi/9)$ is not a root for it. Similarly, $\cos(2\pi/9) \neq \frac{1}{2}$, thus not a root of $(2x+1)$, so we ignore it... so $\cos(2\pi/9)$ must be a root of $g(x) = (8x^3 - 6x + 1)$.

$$g(x+1) = (8x^3 + 24x^2 + 24x + 8) + (-6x - 6) + 1$$
$$= 8x^3 + 24x^2 + 18x + 3$$

let $p = 3$, then $p \mid a_0$, $p \mid a_1$, $p \mid a_2$, $p \nmid a_3$, and $p^2 \nmid a_0$, thus $g(x+1)$ irreducible by Eisensteins Criterion, implying that $g(x) = 8x^3 - 6x + 1$ is irreducible, making $g(x)$ a minimum polynomial with root $\cos(2\pi/9)$ and $\deg(g(x)) = 3$ but we have a Thm (can't remember the name right now) that tells us...

4a. (5 pts) Use straightedge and compass only to construct $\frac{\sqrt{5}}{2}$, given length 1 below. (You ...rem is ...llowed.)

Continued on back

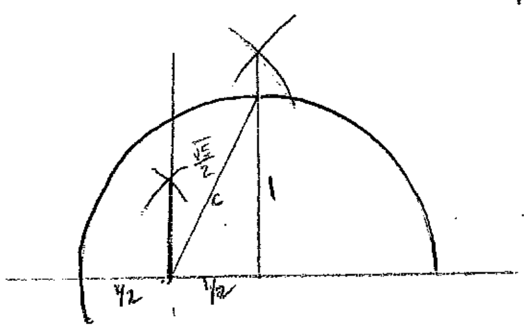
3 continued | ...tells us that $\cos\left(\frac{2\pi}{9}\right)$ is not const'ble since it is the root of a minimal polynomial whose degree isn't a power of 2.

Thus $\cos\left(\frac{2\pi}{9}\right)$ is not const'ble.

And since a regular n-gon is only constructible if $\cos\left(\frac{2\pi}{n}\right)$ is constructible, we have that a regular 9-gon is not constructible since $\cos\left(\frac{2\pi}{9}\right)$ is not constructible.

4a. (5 pts) Use straightedge and compass only to construct $\frac{\sqrt{5}}{2}$, given length 1 below. (You must *do* the construction here; citing a theorem is not allowed.)

1



$1^2 + \left(\frac{1}{2}\right)^2 = c^2$

$1 + \frac{1}{4} = c^2$

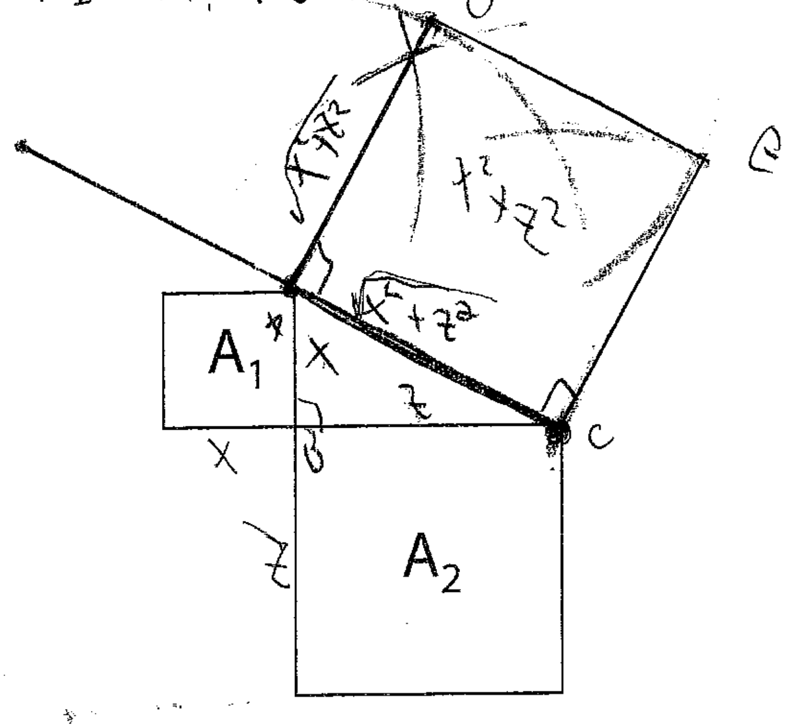
$\frac{5}{4} = c^2$

$\frac{\sqrt{5}}{2} = c$

b. (4 pts) Prove that your construction in part (a) works.

5a. (5 pts) Given squares with areas A_1 and A_2 below, use straightedge and compass only to construct a square of area $A_1 + A_2$. (Angles that look like right angles, are right angles.)

$A_1 + A_2 = X^2 + z^2$



b. (3 pts) Prove that your construction in part (a) above works. (I.e. Prove that the area of the square you constructed equals $A_1 + A_2$.)

Let A be a vertex of square A_1
and B, C be vertices of square A_2
Let $AB = X$ and $BC = z$
$\Rightarrow AC = \sqrt{X^2+z^2}$ by Pythagorean theorem
So $A_1 + A_2 = X^2 + z^2 \Rightarrow$ the length of a side of $A_1 + A_2$ is $\sqrt{x^2+z^2}$, so construct a square with all four sides of length AC

6. (6 pts) Find a factor of 2,251,799,813,685,249 $= 2^{51} + 1$.

$2^{51} + 1 = (2^3)^{17} + 1 = (2^3+1)\left((2^3)^{16} - (2^3)^{15} + (2^3)^{14} + \dots - (2^3) + 1\right)$

$= (9)\left((2^3)^{16} - (2^3)^{15} + (2^3)^{14} + \dots - (2^3)\right)$

So 9 is a factor of $2^{51}+1$

14

7. (5 pts each) For which of the following values of n is the regular $n -$ gon constructible? (In each case you must explain how you reach your conclusion.)

a. $n = 170$

$2^{2^2} + 1$

$170 = 5 \cdot 2 \cdot 17$
$= 2 \cdot 5 \cdot 17$, 5, 17 are Fermat primes,
which is a prime factorization of a power of 2 and distinct fermat primes, 170-gon is constructible.

b. $n = 250$

$= 5^3 \cdot 2$
It is a prime factorization of 3 non distinct fermat primes, so a 250-gon is not constructible.

c. $n = 64$

$64 = 2^6$
It has a prime factorization of a power of 2 only, so a 64-gon is constructible.

d. $n = 130$

$= 13 \cdot 2 \cdot 5$
13 is not a fermat prime, so a 130-gon is not constructible.

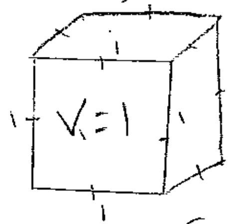
8. (6 pts) Is $\sqrt{\pi} - 1$ constructible with straightedge and compass only? If it is, do the construction. If it is not, prove it.

Assume $\sqrt{\pi} - 1$ is constructible
Let $x = \sqrt{\pi} - 1$
$\Rightarrow x + 1 = \sqrt{\pi}$
$\Rightarrow \pi = (x+1)^2$
Since x is constructible $\Rightarrow x+1$ is constructible
$\Rightarrow (x+1)^2$ is constructible
$\Rightarrow \pi$ is constructible $\Rightarrow \Leftarrow$
since π is not constructible
$\therefore \sqrt{\pi} - 1$ is not constructible.

26

9. (8 pts) Given a cube of volume 1, can a cube with four times the volume be constructed with straightedge and compass only? (I.e. Can the cube be "quadrupled"?) If it can, do the construction; if it can't, prove it.

4V $\Rightarrow$ each side is $\sqrt[3]{4}$



$x = \sqrt[3]{4} \Rightarrow x^3 = 4 \Rightarrow x^3 - 4 = 0$

So $\sqrt[3]{4}$ is a root of $f(x) = x^3 - 4$.

$f(x+1) = x^3 + 3x^2 + 3x - 3$, Using Eisenstein's criterion, with $p = 3$, $3 \nmid 1$, $3 \mid 3$, & $3 \mid -3$, so $f(x)$ is minimal. Since $\deg(f(x)) = 3 \neq$ power of 2, $\sqrt[3]{4}$ is not constructible, thus the cube with $V=4$ is not constr.

10. (9 pts) Given that 1, α, and β are constructible numbers whose lengths are given below, prove that the number $\frac{\beta}{\alpha}$ is constructible with straightedge and compass only. (You must *do* the construction here and explain why it works; citing a theorem is not allowed.)

1

α

β

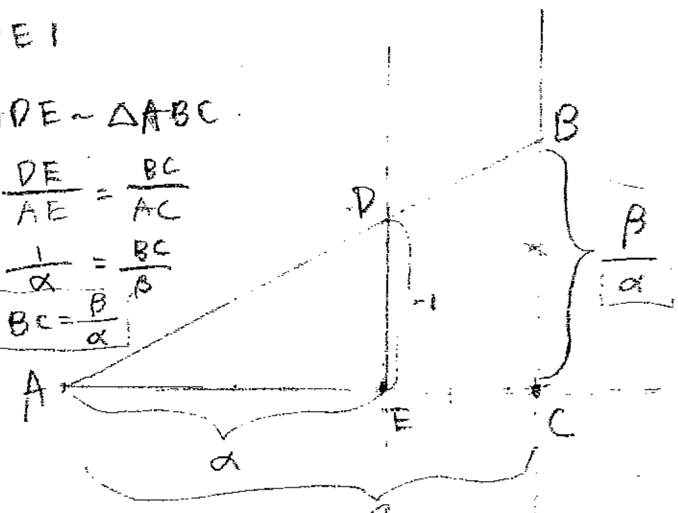
AE $= \alpha$
AC $= \beta$
DE 1

$\triangle ADE \sim \triangle ABC$

So, $\frac{DE}{AE} = \frac{BC}{AC}$

$\frac{1}{\alpha} = \frac{BC}{\beta}$

$\boxed{BC = \frac{\beta}{\alpha}}$



11. (6 pts) Can a 90 degree angle be trisected using straightedge and compass only? Explain why it can, or prove that it can't.

Yes, it can by constructing an equilateral triangle with one vertex touching the 90° angle, & then bisecting the 60° angle in the equilateral triangle.

4