

## Deductive Geometry

*Deductive geometry* is the art of deriving new geometric facts from previously-known facts by using logical reasoning. In elementary school, many geometric facts are introduced by folding, cutting, or measuring exercises, not by logical deduction. But as we have seen, fifth and sixth grade students are already practicing — and enjoying — deductive reasoning as they solve unknown angle problems.

In geometry, a written logical argument is called a *proof*. Section 4.1 introduces one type of proof: “unknown angle proofs”. Unknown angle proofs are natural continuations of students’ experience in solving unknown angle problems; the transition is a small step that requires no new concepts. Indeed, as you will see, unknown angle proofs are almost identical to the “Teacher’s Solutions” that you wrote in the previous chapter!

Section 4.2 describes how congruent triangles are introduced in middle school. Congruence is a powerful geometric tool that opens a door to new aspects of geometry; some of this is covered in Sections 4.3 and 4.4. These sections also describe how the facts about triangles and quadrilaterals that students learned in grades 5 and 6 are revisited at a higher level in middle school.

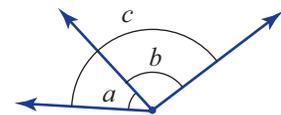
In this chapter we reach the last stage in the preparation of students for high school geometry. As you read and do problems, think about how these problems are part of a story line that goes back to learning to measure angles in grade 4 and learning to measure lengths in grade 1. You will be teaching part of this story, and it is important to know how it unfolds.

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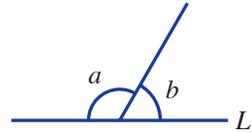
### Background Knowledge

Here is a list of the geometric facts at our disposal at this point. These facts will be used in the examples and homework problems in this chapter. Several additional facts will be added to this list in Section 4.2.

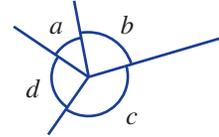
- The measures of adjacent angles add.  
( $c = a + b$ .)  
Abbreviation: **∠s add.**



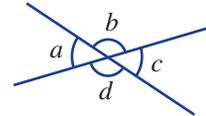
- The sum of adjacent angles on a straight line is  $180^\circ$ .  
(If  $L$  is a line then  $a + b = 180^\circ$ .)  
Abbreviation:  **$\angle$ s on a line.**



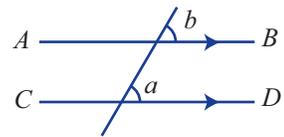
- The sum of adjacent angles around a point is  $360^\circ$ .  
( $a + b + c + d = 360^\circ$ .)  
Abbreviation:  **$\angle$ s at a pt.**



- Vertically opposite angles are equal.  
(At the intersection of two straight lines,  $a = c$  and  $b = d$ .)  
Abbreviation: **vert.  $\angle$ s.**

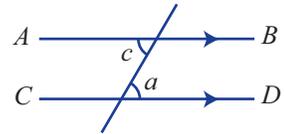


- When a transversal intersects parallel lines, corresponding angles are equal.  
(If  $\overline{AB} \parallel \overline{CD}$  then  $a = b$ .)  
Abbreviation: **corr.  $\angle$ s,  $\overline{AB} \parallel \overline{CD}$ .**



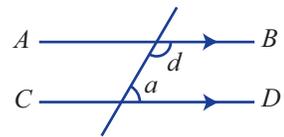
- Conversely, if  $a = b$  then  $\overline{AB} \parallel \overline{CD}$ .  
Abbreviation: **corr.  $\angle$ s converse.**

- When a transversal intersects parallel lines, alternate interior angles are equal.  
(If  $\overline{AB} \parallel \overline{CD}$  then  $a = c$ .)  
Abbreviation: **alt.  $\angle$ s,  $\overline{AB} \parallel \overline{CD}$ .**



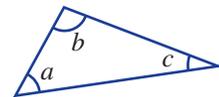
- Conversely, if  $a = c$  then  $\overline{AB} \parallel \overline{CD}$ .  
Abbreviation: **alt.  $\angle$ s converse.**

- When a transversal intersects parallel lines, interior angles on the same side of the transversal are supplementary.  
(If  $\overline{AB} \parallel \overline{CD}$  then  $a + d = 180^\circ$ .)  
Abbreviation: **int.  $\angle$ s,  $\overline{AB} \parallel \overline{CD}$ .**



- Conversely, if  $a + d = 180$  then  $\overline{AB} \parallel \overline{CD}$ .  
Abbreviation: **int.  $\angle$ s converse.**

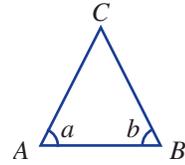
- The angle sum of any triangle is  $180^\circ$ . (\*)  
( $a + b + c = 180^\circ$ .)  
Abbreviation:  **$\angle$  sum of  $\Delta$ .**



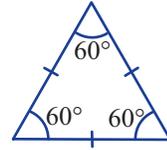
- Each exterior angle of a triangle is the sum of the opposite interior angles. (\*)  
( $e = a + b$ .)  
Abbreviation: **ext.  $\angle$  of  $\Delta$ .**



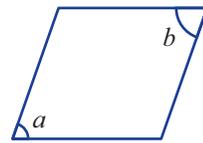
- Base angles of an isosceles triangle are equal. (\*)  
(If  $AC = BC$  then  $a = b$ .)  
Abbreviation: **base  $\angle$ s of isos.  $\Delta$** .



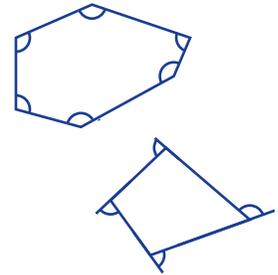
- Each interior angle of an equilateral triangle is  $60^\circ$ . (\*)  
Abbreviation: **equilat.  $\Delta$** .



- Opposite angles in a parallelogram are equal. (\*)  
( $a = b$ .)  
Abbreviation: **opp.  $\angle$ s //ogram**.



- The sum of the interior angles of an  $n$ -gon is  $(n - 2) \cdot 180^\circ$ .  
Abbreviation:  **$\angle$  sum of  $n$ -gon**.
- The sum of the exterior angles of a convex  $n$ -gon is  $360^\circ$ .  
Abbreviation: **ext.  $\angle$ s of cx.  $n$ -gon**.



(\*) The starred facts were established by fifth grade classroom demonstrations. Later in this chapter we will give deductive proofs for them.

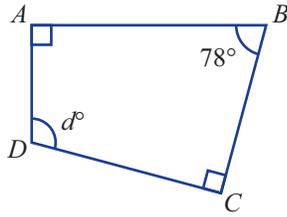
## 4.1 Unknown Angle Proofs

The previous chapter introduced the idea of a “Teacher’s Solution”, which you then used for homework solutions. This specific format is designed to help make you aware of all the aspects of a solution that must be communicated to students and to emphasize that this communication requires only a few words. Many problems in the New Elementary Math textbooks lead students to write teacher solutions themselves.

The Teacher’s Solution format serves another purpose: it helps pave the way for proofs. In fact, a Teacher’s Solution can be made into a proof by simply changing one specific measurement (such as  $31^\circ$ ) into a measurement specified by a letter (such as  $x^\circ$ ). Beyond that, there are only stylistic differences between unknown angle problems and unknown angle proofs.

Examples 1.1 and 1.2 illustrate the transition from unknown angle problems to unknown angle proofs. This section also introduces a format for writing simple proofs in a manner that is almost identical to the Teacher’s Solutions you have done already.

**EXAMPLE 1.1.** In the figure, angles  $A$  and  $C$  are right angles and angle  $B$  is  $78^\circ$ . Find  $d$ .

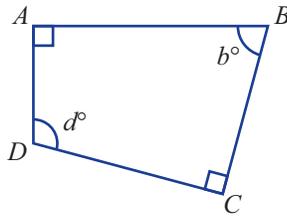


**Teacher's Solution:**

$$\begin{aligned} 90 + 78 + 90 + d &= 360 && \angle \text{ sum in 4-gon} \\ 180 + 78 + d &= 360 \\ 78 + d &= 180 \\ \therefore d &= 102. \end{aligned}$$

Example 1.1 is a fact about one particular shape. But if we replace the specific measurement  $78^\circ$  by an unspecified angle measure  $b^\circ$ , then the identical reasoning yields a general fact about quadrilaterals with two  $90^\circ$  interior angles.

**EXAMPLE 1.2.** In the figure, angles  $A$  and  $C$  are right angles. Prove that  $d = 180 - b$ .

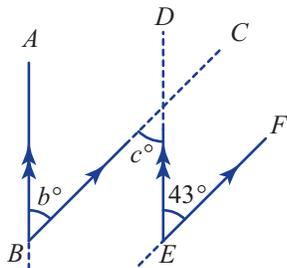
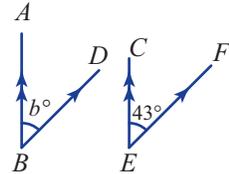


*Proof.*

$$\begin{aligned} 90 + b + 90 + d &= 360 && \angle \text{ sum in 4-gon} \\ 180 + b + d &= 360 \\ b + d &= 180 \\ \therefore d &= 180 - b. \end{aligned}$$

Notice the distinction between the above examples. Example 1.1 is an unknown angle problem because its answer is a number:  $d = 102$  is the number of degrees for the unknown angle. We call Example 1.2 an **unknown angle proof** because the conclusion  $d = 180 - b$  is a relationship between angles whose size is not specified.

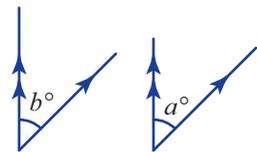
**EXAMPLE 1.3.** In the figure,  $\overline{AB} \parallel \overline{EC}$  and  $\overline{BD} \parallel \overline{EF}$ . Find  $b$ .



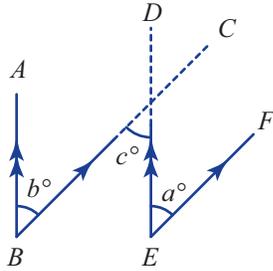
**Teacher's Solution:** Extend the lines as shown.  
Mark angle  $c$  as shown.

$$\begin{aligned} c &= 43 && \text{alt. } \angle\text{s, } \overline{BC} \parallel \overline{EF} \\ b &= c && \text{alt. } \angle\text{s, } \overline{BA} \parallel \overline{ED} \\ \therefore b &= 43. \end{aligned}$$

There is nothing special about the number 43. The same reasoning shows that  $b = a$  in the picture on the right. The proof below is a Teacher's Solution with two embellishments. First, it is "launched" by a preamble that states, in very few words, what we are assuming as known and what we wish to show. Second, the solution involves two auxiliary lines, as we explain to the reader on a line labeled "construction".



Given:  $\overline{AB} \parallel \overline{ED}$  and  $\overline{BC} \parallel \overline{EF}$ .  
 To prove:  $b = a$ .

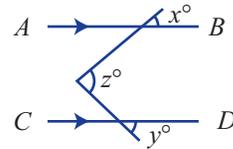


Construction: Extend sides  $\overline{BC}$  and  $\overline{ED}$ .  
 Mark angle  $c$  as shown.

Proof.  $c = a$  alt.  $\angle$ s,  $\overline{BC} \parallel \overline{EF}$   
 $b = c$  alt.  $\angle$ s,  $\overline{BA} \parallel \overline{ED}$   
 $\therefore b = a$ .

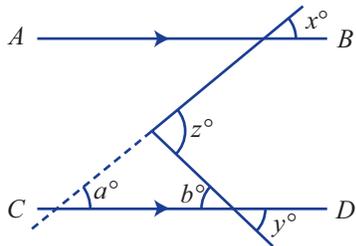
We have just turned Example 1.3 into a proof. The proof requires no additional effort!

**EXAMPLE 1.4.** In the figure,  $\overline{AB} \parallel \overline{CD}$ .  
 Prove that  $z = x + y$ .



You have seen this problem before: it is almost identical to Example 2.3 on page 65. The previous version was an unknown angle problem: two of the three angles  $x$ ,  $y$ , and  $z$  were given (one was  $37^\circ$  and another was  $75^\circ$ ) and the problem was to find the third. The version above leaves  $x$ ,  $y$ , and  $z$  unspecified and the problem is to prove that they are related.

Just as before, the proof requires a construction, and several different constructions will work. Here is one proof.



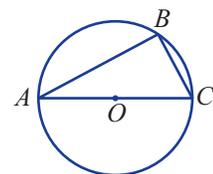
Given:  $\overline{AB} \parallel \overline{CD}$ .  
 To prove:  $z = x + y$ .

Construction: Extend line as shown.  
 Mark angles  $a$  and  $b$  as shown.

Proof.  $a = x$  corr.  $\angle$ s,  $\overline{AB} \parallel \overline{CD}$   
 $b = y$  vert.  $\angle$ s  
 $z = a + b$  ext.  $\angle$ s of a  $\Delta$   
 $\therefore z = x + y$ .

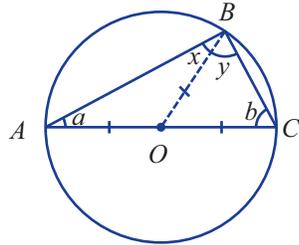
A polygon whose vertices lie on a circle is said to be **inscribed** in the circle. The next example proves a famous fact about inscribed triangles.

**EXAMPLE 1.5.** Any inscribed triangle with a side passing through the center of the circle is a right triangle.



Given:  $\triangle ABC$  is inscribed in a circle.  
 $\overline{AC}$  contains center  $O$ .

To prove:  $\angle B = 90^\circ$ .



Construction: Draw segment  $\overline{OB}$ .

Mark angles as shown.

Proof.  $OA = OB = OC$  def. of circle  
 (each is a radius)  
 $\therefore \triangle OAB$  and  $\triangle OBC$  are isosceles  $\triangle$ s  
 $a = x$  base  $\angle$ s of isos.  $\triangle$   
 $b = y$  base  $\angle$ s of isos.  $\triangle$   
 $x + y + a + b = 180^\circ$   $\angle$  sum of  $\triangle ABC$   
 $x + y + x + y = 180^\circ$   
 $2(x + y) = 180^\circ$   
 $\therefore \angle B = x + y = 90^\circ$   $\angle$ s add.

There are two standard ways for writing angles in geometric figures. One possibility is to label angles in figures with a letter and a degree sign, as in  $x^\circ$ . Then  $x$  stands for a number, for example,  $x$  might be 42. The alternative is to label angles in figures with a letter only, as was done in Example 1.5. Then  $x$  stands for a quantity,  $x$  might be 42 degrees. Both are correct *provided the notation is consistent within each problem*. In particular, equations should be written so that every term is a number, or every term is a quantity.

**EXERCISE 1.6.** In which of the solved examples in this section do letters stand for numbers? In which do they stand for quantities?

For elementary students, it is best to make the units — the degree signs — plainly visible in the figure. But for unknown angle proofs, the presentation is often clearer when letters stand for quantities. Accordingly, we scrupulously included degree signs in Chapters 1, 2 and 3, but we will now begin using both notations freely.

### The “Elementary Proof” Format

Proofs are exercises for students! The challenge — and the fun — lies in figuring out the sequence of steps that take you to the desired conclusion. But, as with multi-step problems, it is easy for students to get confused unless they *systematically record their steps in writing*. Students who have learned a specific format for recording steps can devote more attention to geometric thinking and less to organizing their writing. A uniform format also facilitates classroom discussions and makes it easier for teachers to read and evaluate student work.

Which format is best? There is no single answer, but several principles are important. Writing proofs shouldn't be a chore, so the chosen format should be simple, minimal, and natural. It is useful to include a preamble stating the hypotheses and the goal, because this *helps students clearly understand the task* before starting. A clearly-marked picture obviously helps students work out the solution. Finally, the completed proof should make the reasoning clear to the reader.

And who is this mysterious reader? At all levels of K-12 mathematics, students write for two readers: *themselves and their teacher*. For students, clear writing aids clear thinking. For

teachers, asking students to use a simple, clear format is a matter of self-interest: it makes student work easier to read. In fact, this applies to everything written by mathematics students.

In this book, we will adopt the format used in the examples in this section. We will call a proof written in this format an **Elementary Proof**. You will be expected to use this in all homework problems that ask for an Elementary Proof. The following template shows the features of an Elementary Proof.

**Elementary Proof**

*Preamble states the given information and the goal.*

Given:  $\overline{AD} \parallel \overline{CE}$  and  $\overline{AB} \perp \overline{BC}$ .  
 To Prove:  $x + y = 90^\circ$ .

Construction: Extend  $\overline{AB}$  to meet  $E$  at angle  $z$ .

Proof:  $x = z$                       corr.  $\angle$ s,  $\overline{AD} \parallel \overline{CE}$   
 $z + y = 90^\circ$                     ext.  $\angle$  of  $\Delta$   
 $\therefore x + y = 90^\circ$ .

*Any needed construction is explained before the proof starts.*

*Facts used are recorded using our abbreviations.*  
*Only 1 fact per line.*

*Diagram shows all points, lines and angles used.*

*Last line identical to the "To Prove" line.*

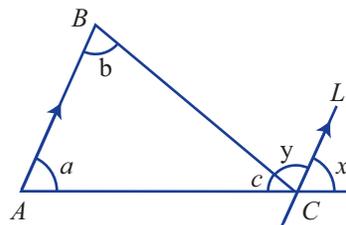
**Hints.** Clear proofs are short and simple. To that end,

- Do not label the two columns “statement” and “reason” (everyone already knows this!).
- Do not include reasons for simple arithmetic and algebra steps.
- To avoid cluttering the picture, label *only* those points, lines and angles used in the proof.

**Establishing Facts Using Proofs**

The next two examples show how two familiar facts about triangles follow from the properties of parallel lines. In fifth grade, students justified the “angle sum of triangle” fact by cutting and re-arranging paper triangles. Middle school students can give a purely geometric argument for this fact. These are very important proofs: learn them.

**THEOREM 1.7.** *In any triangle, the sum of the interior angles is  $180^\circ$ .*



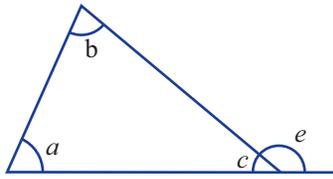
Given:  $\Delta ABC$ .

To prove:  $a + b + c = 180^\circ$ .

Construction: Draw line  $L$  through  $C$  parallel to  $\overline{AB}$ . Mark angles  $x$  and  $y$  as shown.

Proof:  $a = x$                       corr.  $\angle$ s,  $L \parallel \overline{AB}$   
 $b = y$                               alt. int.  $\angle$ s,  $L \parallel \overline{AB}$   
 $c + x + y = 180^\circ$              $\angle$ s on a line  
 $\therefore a + b + c = 180^\circ$ .

**THEOREM 1.8.** Each exterior angle of a triangle is the sum of the opposite interior angles.



Given:  $\triangle ABC$ .

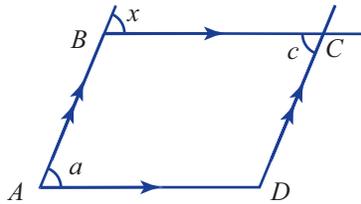
To prove:  $e = a + b$ .

*Proof.*  $e = 180^\circ - c$        $\angle$ s on line  
 $a + b = 180^\circ - c$      $\angle$  sum of  $\triangle$   
 $\therefore e = a + b$ .

**EXERCISE 1.9.** Compare both theorems with the “picture proof” of the same fact described in Section 2.2 of Chapter 2. Are they compatible? Do these proofs elaborate on the picture proofs?

The next proof is different from the fifth-grade paper-folding explanation, but is still easy to understand.

**THEOREM 1.10.** Opposite angles in a parallelogram are equal.



Given:  $ABCD$  is a parallelogram.

To prove:  $a = c$ .

*Construction:* Extend sides  $\overline{AB}$ ,  $\overline{DC}$ , and  $\overline{BC}$ .  
 Mark angle  $x$  as shown.

*Proof.*  $a = x$     corr.  $\angle$ s,  $\overline{BC} \parallel \overline{AD}$   
 $x = c$     alt. int.  $\angle$ s,  $\overline{CD} \parallel \overline{AB}$   
 $\therefore a = c$ .

### Theorems and Proofs in the Classroom

Students often acquire misconceptions about the meaning of the words “theorem” and “proof”. Many believe that a theorem is “a mathematical fact” and a proof is “an explanation of why a fact is true.” This viewpoint embodies a subtle misunderstanding that teachers should try to prevent from taking root.

Theorems are not statements of universal truths. Rather, they are “if-then” statements: *If* certain assumptions are true *then* a stated conclusion is true. The assumptions are of two types: those explicitly stated as hypotheses (after the word *Given* in our format), and a collection of “background knowledge” facts (often not explicitly mentioned) that have already been accepted or proven true. A proof is a sequence of deductive steps that explain how the conclusion follows from the hypothesis. This can be said more compactly as follows:

**DEFINITION 1.11.** A *proof* of a mathematical statement is a detailed explanation of how that statement follows logically from other statements already accepted as true.

A *theorem* is a mathematical statement with a proof.

Theorems are the building blocks of geometry. Once a theorem has been proved, it can be added to the list of background facts and used in subsequent proofs. For example, after proving

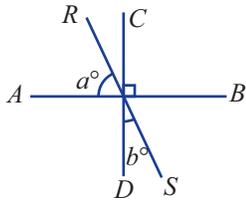
Theorem 1.7 we were able to use it as a step in the proof of Theorem 1.8. In this way one builds, bit by bit, a large body of knowledge.

This structure puts a burden on textbooks and teachers. A proof only makes sense in the context of a particular collection of accepted background facts, and that collection changes grade by grade. Teachers should devote some time to clarifying the already-accepted background facts for their grade.

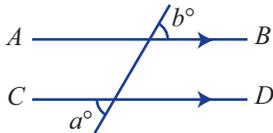
### Homework Set 13

Following the format described in this section, give an “Elementary Proof” of each statement.

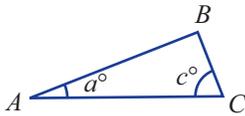
1. In the figure, prove that  $a + b = 90$ .



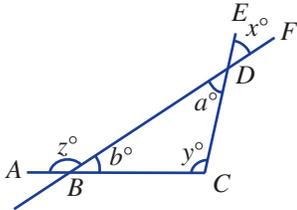
2. In the figure,  $\overline{AB} \parallel \overline{CD}$ . Prove that  $a = b$ .



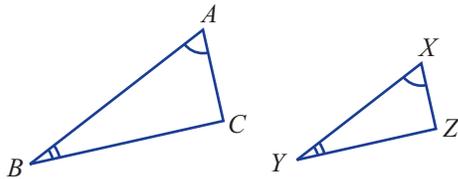
3. In the figure,  $\angle B = 90^\circ$ . Prove that  $a + c = 90$ .



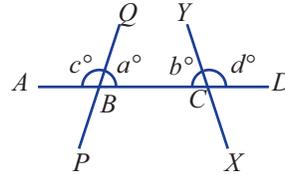
4. In the figure, prove that  $z = x + y$ .



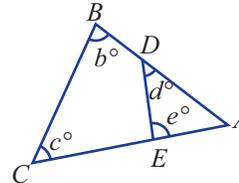
5. In the figure,  $\angle A = \angle X$ ,  $\angle B = \angle Y$ . Prove that  $\angle C = \angle Z$ .



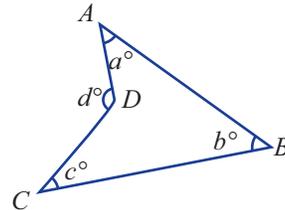
6. In the figure,  $a = b$ . Prove that  $c = d$ .



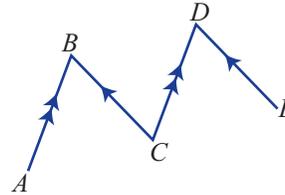
7. In the figure, prove that  $d + e = b + c$ .



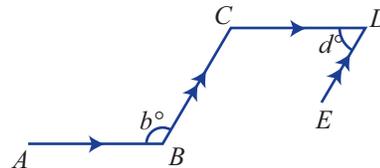
8. In the figure, prove that  $d = a + b + c$ .



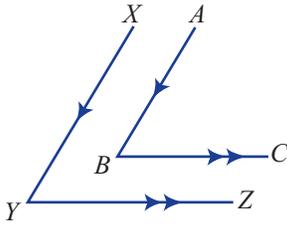
9. In the figure,  $\overline{AB} \parallel \overline{CD}$  and  $\overline{CB} \parallel \overline{ED}$ . Prove that  $\angle ABC = \angle CDE$ .



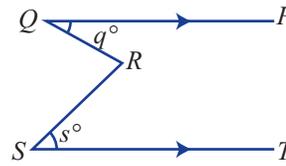
10. In the figure,  $\overline{AB} \parallel \overline{CD}$  and  $\overline{BC} \parallel \overline{ED}$ . Prove that  $b + d = 180$ .



11. In the figure,  $\overline{AB} \parallel \overline{XY}$  and  $\overline{BC} \parallel \overline{YZ}$ . Prove that  $\angle ABC = \angle XYZ$ .



12. In the figure,  $\overline{PQ} \parallel \overline{TS}$ . Prove that  $\angle QRS = q + s$ .



## 4.2 Congruent Triangles

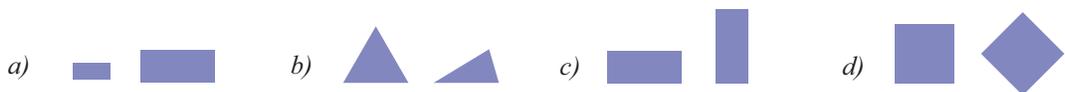
Two line segments are called congruent if they have equal lengths. Two angles are congruent if they have equal measures. Similarly, two triangles are called congruent if their sides have the same lengths and their angles have the same measures. The first part of this section explores how curricula build up to this notion of congruent triangles. The second part focuses on the criteria that ensure that two triangles are congruent.

### A Curriculum Sequence

**Early Grades.** In geometry, figures that are duplicates – exact copies of one another – are called *congruent*. Duplicate figures can be drawn in different positions and orientations. Thus, to check whether two figures are congruent one must realign them to see if they match.

Young children are taught to match shapes, first with figures made of cardboard or thin plastic, then visually with pictures. The phrase “same size and same shape” is often used to mean “congruent”. The grade 1 Primary Mathematics books contain exercises like this:

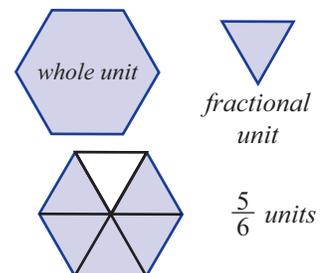
*Do the figures have the same size and same shape?*



These are exercises in visualization only; students are not expected to make measurements. The goal is to convey the idea that matching requires first sliding and rotating, and then comparing lengths and angles. The term congruence is not used, but the seed of the idea is planted.

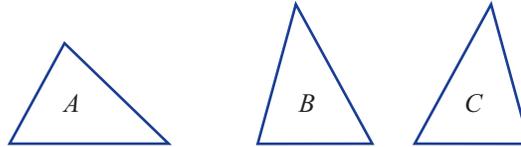
**Teaching comment:** Children may misinterpret the term “same shape” to mean “shapes with the same name”. But the word “shape” refers to the angles and proportions in a figure, not just its name. Thus, the figures in b) above are both triangles, but do not have the same shape.

Matching exercises also introduce ideas that are used later for defining area and for modeling fractions. For example, a regular hexagon, decomposed into 6 congruent triangles, can be used to illustrate fractions whose denominator is 6.

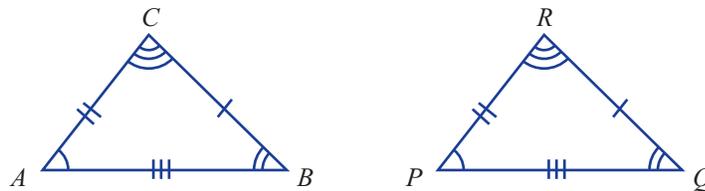


**Middle School Introduction.** The Primary Mathematics curriculum introduces congruence in grade 7 (some curricula start as early as grade 5). One approach starts with see-through tracing paper or overhead transparency sheets as in the following exercise.

**EXERCISE 2.1.** Trace triangle  $A$  on a transparent sheet and lay it over figures  $B$  and  $C$ . Can you make them match? Flip the sheet over and try again.

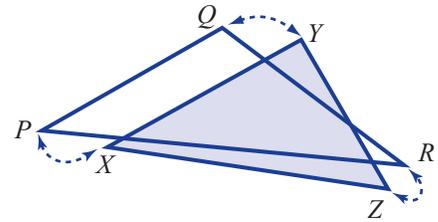


This exercise makes the idea of “same size, same shape” more precise, yet it still relies on visual matching. But notice that “matching” can be described purely in terms of the 3 side lengths and 3 angle measures: two triangles match if they can be aligned so that *all six of these measurements exactly agree*. In diagrams, such matching can be shown by marking the triangles as in the figure below.



But it is more efficient to use symbols. This requires (i) a symbolic way of describing how to align two triangles, and (ii) a definition for what is meant by “exactly match”.

To align triangles, one pairs up their vertices. Such a pairing is called a correspondence. More precisely, a **correspondence** between two triangles pairs each vertex of one triangle with one and only one vertex of the other. The figure shows triangles  $\triangle PQR$  and  $\triangle XYZ$  aligned in a way that suggests the correspondence  $P \leftrightarrow X$ ,  $Q \leftrightarrow Y$ , and  $R \leftrightarrow Z$ .

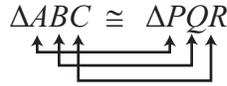


Once a correspondence is chosen, we can talk about *corresponding sides* and *corresponding angles*. For the correspondence  $P \leftrightarrow X$ ,  $Q \leftrightarrow Y$ , and  $R \leftrightarrow Z$ ,  $\angle P$  corresponds to  $\angle X$ , side  $\overline{QR}$  corresponds to side  $\overline{YZ}$ , etc. In this way we can compare the six measurements of two triangles, even if they don’t have the same shape or size. When the corresponding measurements are equal, the correspondence gives us a precise way of stating that the triangles are exact copies of each other.

**DEFINITION 2.2.** Two triangles are **congruent** if, under some correspondence,

- all pairs of corresponding sides are equal, and
- all pairs of corresponding angles are equal.

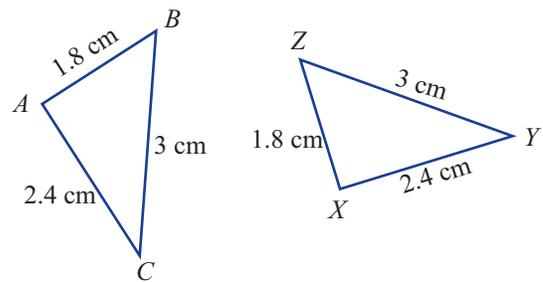
To indicate that two triangles are congruent we use the symbol  $\cong$  and write the vertices of the triangles in corresponding order. Thus  $\triangle ABC \cong \triangle PQR$  means that  $\triangle ABC$  and  $\triangle PQR$  are congruent under the correspondence  $A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$ . To see the correspondence, one can visualize arrows pairing up the vertices in order.



Here is a grade 7 exercise that helps students move from checking congruence visually to thinking about the measurements of corresponding parts:

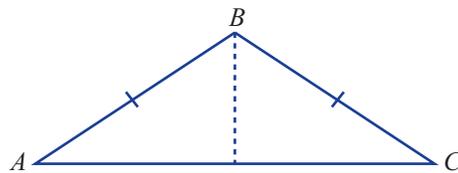
**EXERCISE 2.3.** *These triangles are congruent. Fill in the blanks by matching side lengths and estimating angle measures.*

- $\overline{AB}$  corresponds to  $\underline{\hspace{1cm}}$
- $\overline{BC}$  corresponds to  $\underline{\hspace{1cm}}$
- $\underline{\hspace{1cm}}$  corresponds to  $\overline{XY}$
- $\angle A$  corresponds to  $\angle \underline{\hspace{1cm}}$
- $\angle B$  corresponds to  $\angle \underline{\hspace{1cm}}$
- $\angle \underline{\hspace{1cm}}$  corresponds to  $\angle Y$ .
- $\therefore \triangle ABC \cong \triangle \underline{\hspace{1cm}}$ .



**Hint:** In such a congruence, one can associate each letter with an interior angle, and then the smallest angle of one triangle is paired with the smallest angle of the other, the largest is paired with the largest, and the mid-sized with the mid-sized.

Symmetry can be expressed as a congruence of a triangle *with itself*. For example, the isosceles triangle below has a line of symmetry. Reflecting across the line of symmetry interchanges  $A$  and  $C$  and does not move  $B$ . Thus this reflection is described by the correspondence



$A \leftrightarrow C, C \leftrightarrow A$ , and  $B \leftrightarrow B$ . The reflection takes the 3 sides of the triangle to sides of equal length (since  $AB = BC$ ) and takes the 3 angles to angles of equal measure (since base angles are equal). Consequently, the reflection is a congruence, which we write as  $\triangle ABC \cong \triangle CBA$ .

**EXERCISE 2.4.** *An equilateral triangle  $\triangle ABC$  is congruent to itself in six ways. Write down all six, beginning with  $\triangle ABC \cong \triangle ABC$ . Write “R” next to the congruences that are rotations and “LS” next to those that are reflections across a line of symmetry. It will help to draw pictures.*

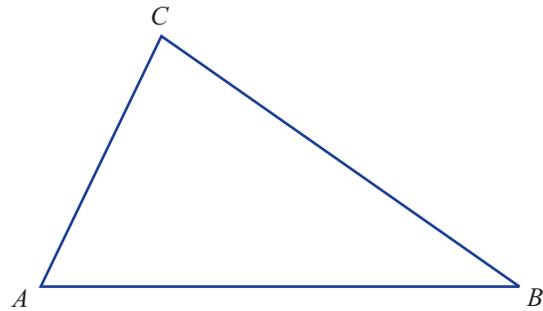
### Congruence Tests for Triangles: a Teaching Sequence

Congruent triangles have six pairs of equal measurements (3 pairs of angles and 3 pairs of side lengths). However, to show that triangles are congruent, it isn't necessary to check all six pairs. Four **congruence criteria** make the task easier. The criteria are often called “tests” and are named by triples: the Side-Side-Side Test, Angle-Side-Angle Test, Side-Angle-Side Test and the Right-Hypotenuse-Side Test. Each test states that two triangles are congruent whenever the named measurements match under some correspondence.

Congruence tests are often introduced in middle school by activities that ask students to construct triangles with specified measurements. The aim is to make the meaning of the tests clear and convince students of their validity.

**Side-Side-Side Test.** Construction 6 on page 53 showed how to use the 3 side-lengths of a triangle to create a duplicate triangle with the same side-lengths. To verify that the two triangles are congruent, one must also know that corresponding angles have equal measure.

**EXERCISE 2.5.** *Using a compass and a straightedge, copy  $\triangle ABC$  onto a blank paper as in Construction 6 on page 53. Then make the following checks:*



**First Check:** *Measure the angles in the copied triangle with a protractor. Are they equal to the corresponding angles in  $\triangle ABC$ ? Are the two triangles congruent?*

**Second Check:** *Cut out your triangle and place it on top of the original triangle in this book. Align the sides. Are all angle measures equal? Are the triangles congruent?*

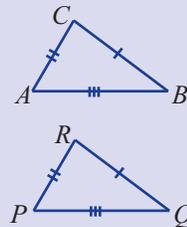
In Exercise 2.5, both checking methods refer directly to the definition of congruence — both require verifying that all corresponding angles and sides are equal. The teaching goal at this stage is to make students explicitly aware of the definition: congruence means 6 matching measurements.

#### Side-Side-Side Test

Two triangles are congruent if corresponding sides are equal under some correspondence, i.e.,

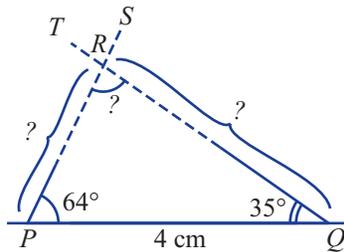
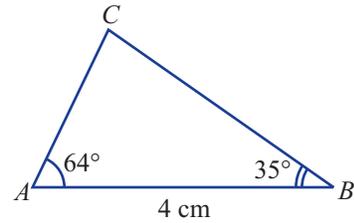
If  $AB = PQ$ ,  $BC = QR$ ,  $CA = RP$ , then  $\triangle ABC \cong \triangle PQR$ .

(Abbreviation: **SSS**.)



**Angle-Side-Angle Test.** In the next example, students duplicate a triangle by copying two angles and the side between them.

**EXERCISE 2.6.** Using a straightedge and protractor, copy  $\triangle ABC$  onto a blank paper as described below. Then make the two checks stated beneath the picture.



*Ruler and Protractor Construction:*

- Draw  $\overline{PQ}$  of length  $AB$ .
- Draw ray  $\overrightarrow{PS}$  so  $\angle SPQ = 64^\circ$ .
- Draw ray  $\overrightarrow{QT}$  so that  $\angle TQP = 35^\circ$ , as shown.
- Mark  $R$  where the two rays intersect and draw  $\overline{PR}$  and  $\overline{QR}$ .

**First Check:** Use a protractor to measure  $\angle R$ ; is it equal to  $\angle C$ ? Use your compass to compare lengths  $QR$  and  $PR$ ; are they equal to  $BC$  and  $AC$ ? Are the two triangles congruent?

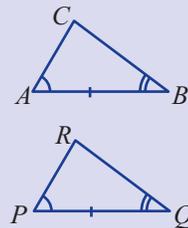
**Second Check:** Cut out your triangle and place it on top of  $\triangle ABC$ . Align the sides. Are all angles and side lengths equal? Are the two triangles congruent?

**Angle-Side-Angle Test**

Two triangles are congruent if two pairs of corresponding angles and their included sides are equal, i.e.,

If  $\angle A = \angle P$ ,  $\angle B = \angle Q$ ,  $AB = PQ$ , then  $\triangle ABC \cong \triangle PQR$ .

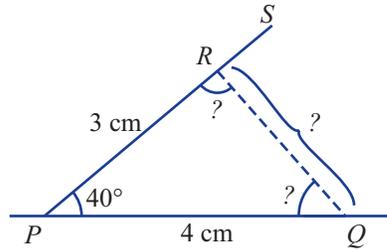
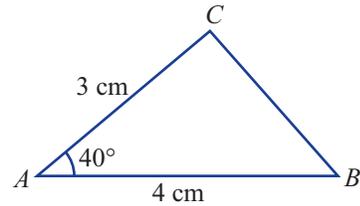
(Abbreviation: **ASA**.)



The ASA Test should really be called the “two angles and a side” test. After all, if two pairs of angles are equal, then the third pair is also equal (because the angles of a triangle total  $180^\circ$ ). Consequently, each side lies between two pairs of corresponding angles, and we can apply the ASA Test. Some books distinguish between the ASA and AAS conditions, but we will use the term “ASA Test” for both.

**Side-Angle-Side Test.** In the next exercise, students copy a side, the included angle and side of a triangle to create a copy of  $\triangle ABC$  above.

**EXERCISE 2.7.** Using a ruler and protractor, copy  $\triangle ABC$  onto a blank paper using the construction below and make the check described.



*Ruler and Protractor Construction:*

- Draw a segment  $\overline{PQ}$  of length 4 cm.
- Draw ray  $\overrightarrow{PS}$  with  $\angle SPQ = 40^\circ$ .
- Mark point  $R$  on  $\overrightarrow{PS}$  with  $PR = 3$  cm.
- Draw  $\overline{RQ}$ .

**First Check:** Measure angles  $\angle Q$  and  $\angle R$ ; are they equal to  $\angle B$  and  $\angle C$ ? Use your compass to compare the lengths  $QR$  and  $BC$ ; are they equal? What do you conclude about the two triangles?

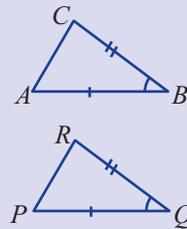
**Second Check:** Cut out your triangle and place it on top of  $\triangle ABC$ . Align the sides. Are the two triangles congruent?

**Side-Angle-Side Test**

Two triangles are congruent if two pairs of corresponding sides and their included angle are equal, i.e.,

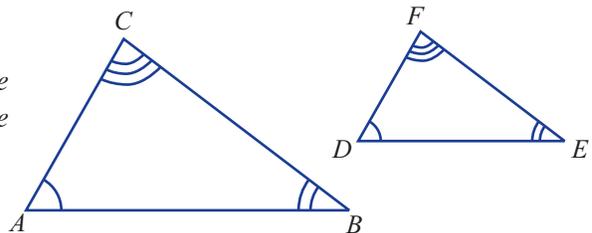
If  $AB = PQ$ ,  $AC = PR$ ,  $\angle A = \angle P$ , then  $\triangle ABC \cong \triangle PQR$ .

(Abbreviation: **SAS**.)

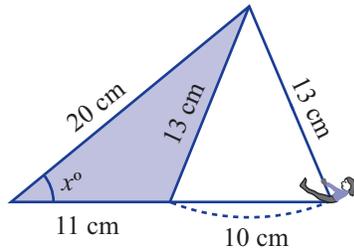


Some sets of three equal corresponding measurements do not guarantee congruence. In particular, triangles with three matching angles needn't be congruent.

*These triangles have the same Angle-Angle-Angle data but are not congruent.*



Likewise, triangles with one pair of equal corresponding angles and two pairs of equal corresponding sides needn't be congruent, as the following "swinging girl" picture shows.

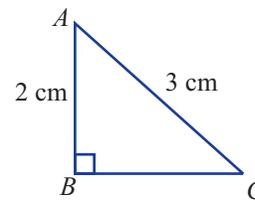


The shaded triangle (base 11 cm) and the large triangle (base 21 cm) have the same Angle-Side-Side data but are not congruent.

The fact that Angle-Side-Side measurements do not determine a unique triangle is easily said: *there is no A.S.S. test*. This phrasing is not recommended for classroom use. Typically, textbooks tell students that "A.A.A. and S.S.A. do not confirm congruence of triangles."

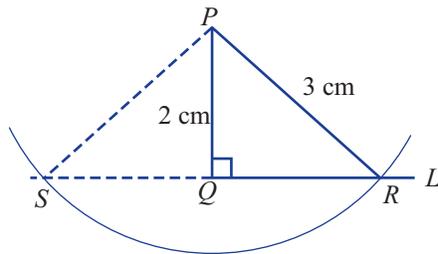
**Right-Hypotenuse-Leg Test.** For pairs of right triangles, it is enough to compare two pairs of sides. In a right triangle, the side opposite the  $90^\circ$  angle is called the *hypotenuse* and the other sides are called *legs*. The following exercise shows that two right triangles are congruent if they have equal hypotenuses are equal and one leg of equal length.

**EXERCISE 2.8.** Copy the right triangle  $\triangle ABC$  onto some tracing paper using the construction below.



Construction:

- Draw a segment  $\overline{PQ}$  of length 2 cm.
- Draw line  $L$  perpendicular to  $\overline{PQ}$  passing through  $Q$ .
- Draw circle, center  $P$ , radius 3 cm.
- Mark as  $R$  one of the two points where the circle intersects  $L$ .



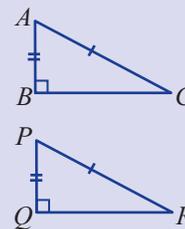
Measure angles  $\angle P$  and  $\angle R$ ; are they equal to  $\angle A$  and  $\angle C$ ? Use your compass to compare the lengths  $QR$  and  $BC$ ; are they equal? What do you conclude about the two triangles?

### Right-Hypotenuse-Leg Test

If two right triangles have hypotenuses of the equal length and a pair of legs with equal length, then the triangles are congruent, i.e.,

If  $\angle B = \angle Q = 90^\circ$ ,  $AC = PR$ ,  $AB = PQ$ , then  $\triangle ABC \cong \triangle PQR$ .

(Abbreviation: **RHL**.)

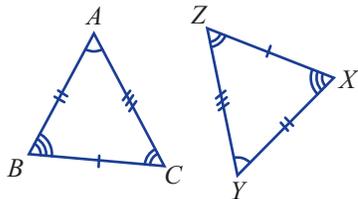


The RHL Test uses two sides and a non-included angle — Angle-Side-Side measurements. In general, for such measurements, the swinging girl picture produces two non-congruent triangles with the same Angle-Side-Side measurements. But the RHL Test is true because, when the angle  $x$  is a right angle, the swinging girl picture becomes the diagram in Exercise 2.8, and the triangles are actually congruent.

**Homework Set 14**

1. Do Exercise 2.3 in this section (just fill in the blanks in your textbook). Then fill in all of the blanks in the exercises below. (On your homework sheet, just state the correspondence for each part.)

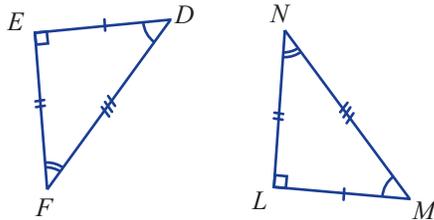
a)



$\angle A = \underline{\hspace{1cm}}$      $\angle B = \underline{\hspace{1cm}}$      $\angle C = \underline{\hspace{1cm}}$   
 $AB = \underline{\hspace{1cm}}$      $BC = \underline{\hspace{1cm}}$      $CA = \underline{\hspace{1cm}}$

Correspondence:  $\triangle ABC \cong \triangle \underline{\hspace{1cm}}$

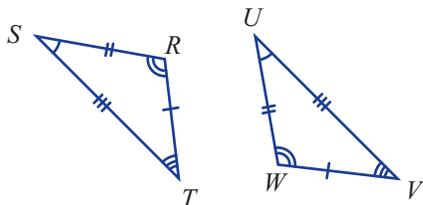
b)



$\angle D = \underline{\hspace{1cm}}$      $\angle E = \underline{\hspace{1cm}}$      $\angle F = \underline{\hspace{1cm}}$   
 $DE = \underline{\hspace{1cm}}$      $EF = \underline{\hspace{1cm}}$      $FD = \underline{\hspace{1cm}}$

Correspondence:  $\triangle DEF \cong \triangle \underline{\hspace{1cm}}$

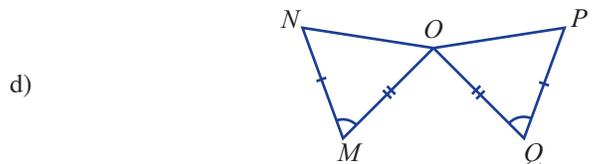
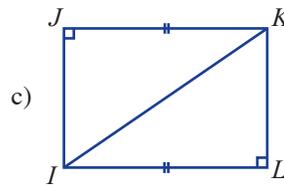
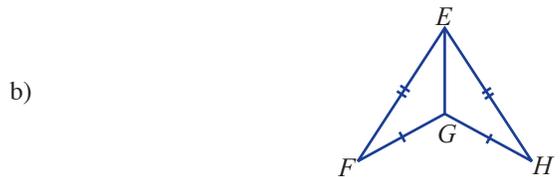
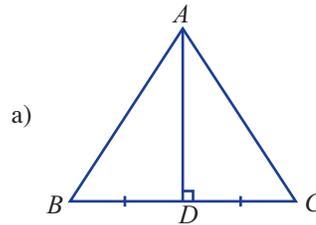
c)

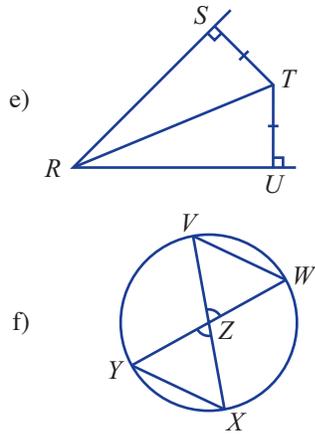


$\angle R = \underline{\hspace{1cm}}$      $\angle S = \underline{\hspace{1cm}}$      $\angle T = \underline{\hspace{1cm}}$   
 $RS = \underline{\hspace{1cm}}$      $ST = \underline{\hspace{1cm}}$      $TR = \underline{\hspace{1cm}}$

Correspondence:  $\triangle RST \cong \triangle \underline{\hspace{1cm}}$

2. Do Exercise 2.5 on page 85 of this section. Answer the “First Check” and “Second Check” questions beneath the picture.
3. Is there an “SSSS Test” for quadrilaterals? That is, is it true that two quadrilateral with four pairs of corresponding sides of equal length are necessarily congruent? Explain and illustrate.
4. Do Exercise 2.6 in this section. First do the construction (in the first step, use your compass to carry the “4 cm” length from your book to your HW paper). Then answer the “First Check” and “Second Check” questions.
5. Do Exercise 2.7 on page 87, this time using *ruler and protractor*. Then answer the “Check” questions.
6. For each figure, name the congruent triangles and state the reason they are congruent (i.e., SSS, ASA, SAS, RHL).

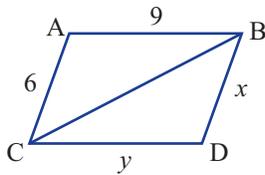




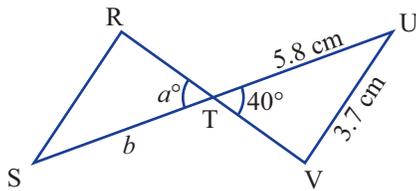
The figure is a circle with center Z.

7. For each figure, find the values of the unknowns.

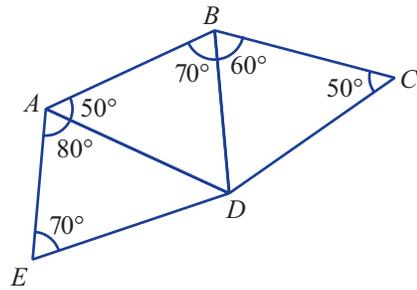
a)  $\triangle ABC \cong \triangle DCB$



b)  $\triangle RST \cong \triangle VUT$



8. In the figure below, which triangles are congruent? Write down each congruence.



9. Given any three positive real numbers,  $a, b$ , and  $c$ , with  $b$  less than 180 and  $c$  less than 180, can you construct a triangle having two interior angles whose measures are  $b$  and  $c$ , respectively, and the included side having measure  $a$ ? Illustrate.

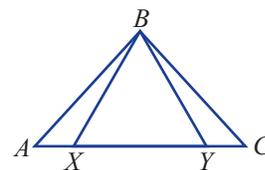
10. Given any three positive real numbers,  $a, b$ , and  $c$ , with  $c$  less than 180, can you construct a triangle having two sides whose measures are  $a$  and  $b$  with the included angle having measure  $c$ ? Illustrate.

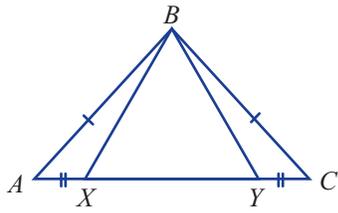
11. Prove or give a counterexample: If two triangles both have interior angle measures equal to  $32^\circ, 100^\circ$  and  $48^\circ$ , and both have a side of length 10 cm, then the triangles are congruent.

### 4.3 Applying Congruences

Once students have learned the triangle tests, they can move beyond unknown angles problems to the next stage in the curriculum: proving facts about side-lengths and angles within figures. This section is an introduction to the simplest proofs. At this point, students appreciate and perhaps enjoy unknown angle problems and short proofs. They are prepared for high school geometry.

**EXAMPLE 3.1.** Let  $\triangle ABC$  be an isosceles triangle with  $AB = BC$ . Let  $X$  and  $Y$  be distinct points on  $\overline{AC}$  such that  $AX = YC$ . Prove that  $\triangle XBY$  is also isosceles.



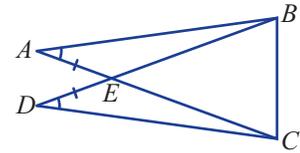


Given:  $AB = BC$  and  $AX = YC$ .  
 To Prove:  $BX = BY$ .

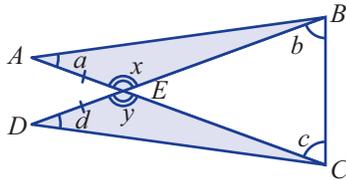
*Proof.*  $AB = BC$  given  
 $\angle A = \angle C$  base  $\angle$ s of isos.  $\Delta$   
 $AX = YC$  given  
 $\therefore \Delta AXB \cong \Delta CYB$  SAS.  
 $\therefore BX = BY$  corr. sides of  $\cong \Delta$ s.

The triangle tests can also be used in unknown angle proofs.

**EXAMPLE 3.2.** In the figure,  $\angle A = \angle D$  and  $AE = DE$ .  
 Prove  $\angle ECB = \angle EBC$ .



To prove this, mark angles as shown and concentrate your attention on the two shaded triangles.



Given:  $a = d$  and  $AE = DE$ .  
 To Prove:  $b = c$ .

*Proof.*  $a = d$  given  
 $AE = DE$  given  
 $x = y$  vert.  $\angle$ s  
 $\therefore \Delta AEB \cong \Delta DEC$  ASA.  
 $\therefore EB = EC$  corr. sides of  $\cong \Delta$ s.  
 $\therefore b = c$  base  $\angle$ s of isos.  $\Delta$ .

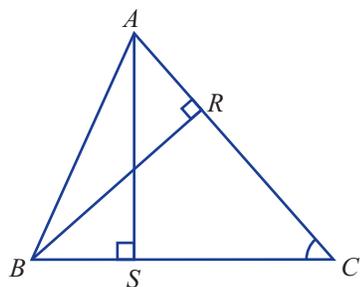
Study Examples 3.1 and 3.2 for a moment. In each proof, Line 4 states that two triangles are congruent by a congruence test, Lines 1-3 are the facts needed to apply the test, and Line 5 is a conclusion based on this congruence. In this chapter, almost all of the proofs will have this format. It is simple, but it has many applications!

false starts  
 in proofs

As students learn to construct proofs, it is easy for them to make a “false start” — their first approach doesn’t work. *This is completely normal!* Proofs are like puzzles: the fun lies in trying different strategies to find a solution. The reward, like the reward in solving a tricky puzzle, is a feeling of accomplishment. In fact, geometric proofs were a common amusement of educated people in the 19th century, just as crossword and sudoku puzzles are today.

The next example is a problem in which it is easy to make a false start. The figure contains several pairs of congruent triangles; which pair should be used in the proof? Try to find a strategy before you look at the proof written below. Here is a strategy that helps: color, highlight or shade the segments that appear in the “Given” and the “To Prove” statements. *Look for a pair of congruent triangles that contain these sides.*

**EXAMPLE 3.3.** In the figure,  $CA = CB$ . Prove that  $AS = BR$ .



Given:  $CA = CB$ .  
To Prove:  $AS = BR$ .

*Proof.* In  $\triangle ACS$  and  $\triangle BCR$ ,  
 $CA = CB$  given  
 $\angle C = \angle C$  common angle  
 $\angle R = \angle S = 90^\circ$  given  
 $\therefore \triangle ACS \cong \triangle BCR$  AAS.  
 $\therefore AS = BR$  corr. sides of  $\cong \Delta$ s.

**EXERCISE 3.4.** This proof used the congruence  $\triangle ACS \cong \triangle BCR$ . Name (without proof) two other pairs of congruent triangles in this figure (use the letter  $T$  to label the intersection point in the middle of the figure).

These examples indicate that there are two levels of congruent triangle proofs: ones that are especially simple because the figure contains only one pair of congruent triangles, and ones in which students must find the appropriate congruent triangles from among several pairs. Careful textbooks (and careful teachers!) provide plenty of practice at the first level before challenging students with problems at the second level. The problems in the homework for this section (HW Set 15) are similar to a well-written eighth grade textbook. As you do these proofs, notice how they are arranged so that they slowly increase in difficulty.

### Proofs for Symmetry Explanations

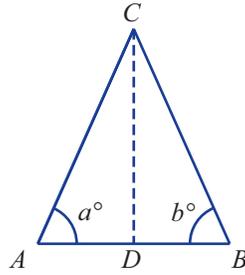
Congruence tests can be used to prove many of the facts that were introduced in fifth and sixth grade using symmetry arguments. We will give three examples. As you will see, the elementary school “folding proofs” contain the key ideas of a complete mathematical proof.

For example, the following proof is just the detailed explanation of the fifth grade picture proof shown on page 62 of Primary Math 5B and on page 44 of this book.

**THEOREM 3.5.** In an isosceles triangle, base angles are congruent.

(Abbreviation: **base  $\angle$ s of isos.  $\Delta$ .**)

Given:  $\triangle ABC$  with  $AC = BC$ .  
To prove:  $a = b$ .

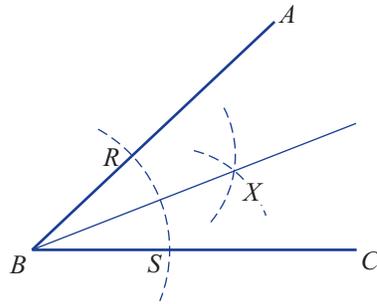
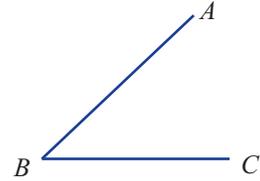


*Construction:* Let  $D$  be the midpoint of  $\overline{AB}$ .  
Draw  $\overline{CD}$ .

*Proof.*  $AC = BC$  given  
 $AD = DB$   $D$  is the midpoint  
 $DC = DC$  common.  
 $\triangle ADC \cong \triangle BDC$  SSS  
 $\therefore a = b$  corr.  $\angle$ s of  $\cong \triangle$ s.

When we first introduced compass and straightedge constructions in Section 2.5, we focused on the steps required and used symmetry to argue that the construction accomplished its intended purpose. We can now replace those symmetry arguments with proofs based on congruent triangles. For example, the construction below shows how to bisect an angle.

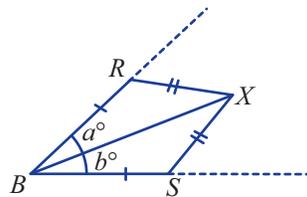
**EXAMPLE 3.6.** *Bisect  $\angle ABC$ .*  
(Abbreviation: **bisect  $\angle$  constr.**)



- Draw circle, center  $B$ , any radius.  
Mark  $R$  and  $S$  where the circle intersects the sides of the angle.
- Draw circle, center  $R$ , radius  $RS$ .  
Draw circle, center  $S$ , radius  $RS$ .
- Mark  $X$  where these two circles intersect.
- Draw  $\overrightarrow{BX}$ ,  $\overrightarrow{RX}$  and  $\overrightarrow{SX}$ .

We claim that this construction works: that  $\overrightarrow{BX}$  bisects  $\angle ABC$ . Symmetry suggests that  $\triangle BRX$  and  $\triangle BSX$  are congruent. Here is a proof:

Given:  $BR = BS$ ,  $RX = SX$ .  
To prove:  $a = b$ .



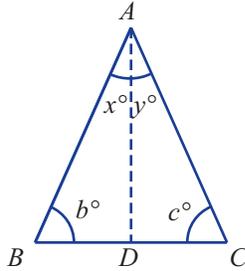
*Proof.*  $BR = BS$  given  
 $RX = SX$  given  
 $BX = BX$  common  
 $\triangle BRX \cong \triangle BSX$  SSS.  
 $\therefore a = b$  corr.  $\angle$ s of  $\cong \triangle$ s.

*All constructions are theorems.* When written down in complete form, each entails a construction followed by a proof that the construction works. We can therefore use constructions

as reasons in proofs. Here is an example:

**THEOREM 3.7.** *If two angles of a triangle are congruent, then the triangle is isosceles.*

(Abbreviation: **base**  $\angle$ s **converse**.)



Given:  $\triangle ABC$  with  $b = c$ .

To Prove:  $AB = AC$ .

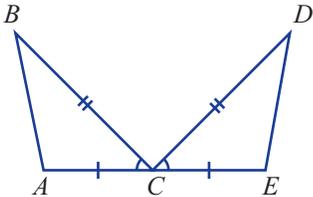
*Construction:* Bisect  $\angle A$  and mark angles  $x$  and  $y$  as shown.

*Proof.*  $b = c$  given  
 $x = y$  bisect  $\angle$  constr.  
 $AD = AD$  common  
 $\therefore \triangle BAD \cong \triangle CAD$  AAS.  
 $\therefore AB = AC$  corr. sides of  $\cong \triangle$ s.

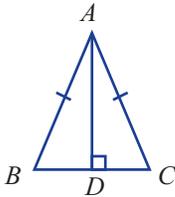
**Homework Set 15**

This homework set gives more practice with proofs using congruent triangles. Please write Elementary Proofs to the following problems. Most of these problems are similar to homework problems found in Japan and Hong Kong in eighth grade.

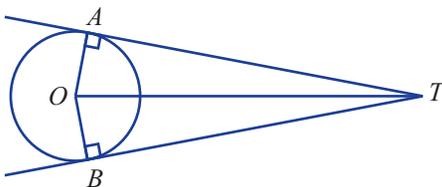
1. In the figure, prove that  $\angle ABC = \angle EDC$ .



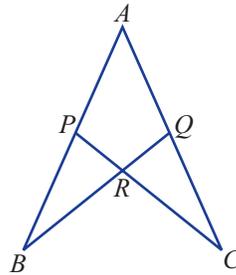
2. In the figure, prove that  $BD = DC$ .



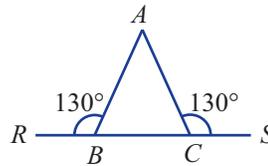
3. In the figure,  $O$  is the center of the circle. Prove that  $TA = TB$ .



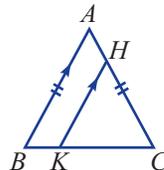
4. In the figure,  $AB = AC$ ,  $AP = AQ$ . Prove that  $BQ = CP$ .



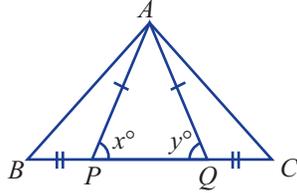
5. In the figure,  $\angle B = \angle C = 130^\circ$ . Prove that  $AB = AC$ .



6. In the figure,  $AB = AC$  and  $\overline{AB} \parallel \overline{HK}$ . Prove that  $HK = HC$ .

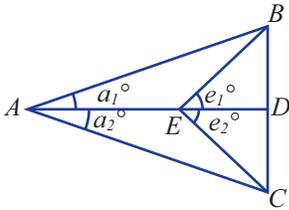


7. In the figure,  $BP = QC$ ,  $AP = AQ$ , and  $x = y$ . Prove that  $AB = AC$ .



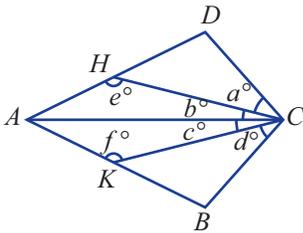
8. In the figure,  $a_1 = a_2$ ,  $e_1 = e_2$ . Prove that

- a)  $\triangle ABE \cong \triangle ACE$ .
- b)  $AB = AC$  and  $\overline{AD} \perp \overline{BC}$ .

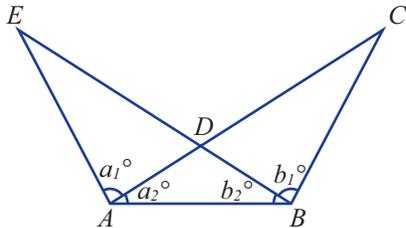


9. In the figure,  $a = d$ ,  $b = c$ , and  $CB = CD$ . Prove that

- a)  $\triangle ADC \cong \triangle ABC$ .
- b)  $e = f$ .

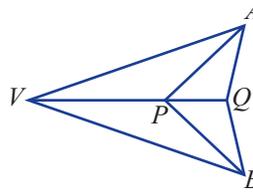


10. In the figure,  $a_1 = b_1$ ,  $a_2 = b_2$ . Prove that  $AC = BE$ .



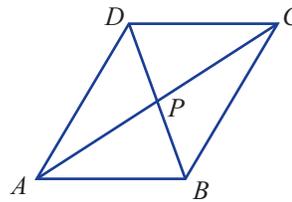
11. In the figure,  $AV = BV$ ,  $AP = BP$ . Prove that

- a)  $\angle APV = \angle BPV$ .
- b)  $AQ = BQ$ .

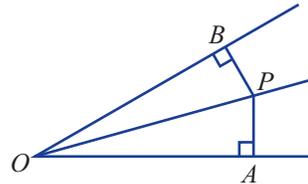


12. In the figure,  $AB = CD$ ,  $BC = AD$ . Prove that

- a)  $\angle ADB = \angle DBC$ .
- b)  $\overline{AC}$  and  $\overline{BD}$  bisect each other.

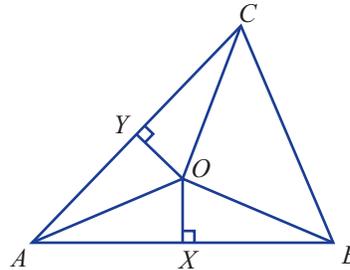


13. In the figure,  $P$  is equidistant from the lines  $OA$  and  $OB$  (i.e.  $PA = PB$ ). Prove that  $OP$  bisects  $\angle AOB$ .



14. In the figure,  $OX$  and  $OY$  are the perpendicular bisectors of  $AB$  and  $AC$  respectively. Prove that

- a)  $\triangle OAX \cong \triangle OBX$ .
- b)  $OA = OB = OC$ .
- c) What can you say about the circle with center  $O$  and radius  $OA$ ? (Hint: Draw it in your book.)



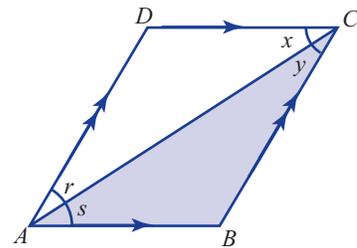
## 4.4 Congruences in Quadrilaterals

You have already seen how the grade 5 (page 68) and grade 6 (page 64–64) Primary Mathematics textbooks present facts about parallelograms. Middle school students can dig deeper and see that the facts they learned about parallelograms and other quadrilaterals are actually consequences of basic facts, namely those listed as “Background Knowledge” at the beginning of this chapter. This section shows how the background knowledge facts can be used to prove various properties of quadrilaterals. It ends with a review of the K-8 curriculum sequence that leads to deductive geometry.

### Parallelograms

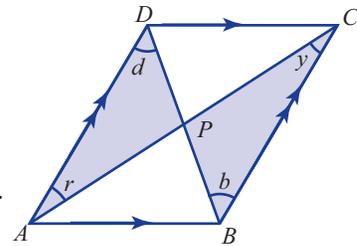
By definition, a parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. Using this definition and congruent triangles, we can prove additional properties of parallelograms, as in the following proofs.

- (1) In  $\triangle ABC$  and  $\triangle CDA$ ,
- |                                     |  |
|-------------------------------------|--|
| $s = x$                             | alt. $\angle$ s, $\overline{AB} \parallel \overline{CD}$ |
| $r = y$                             | alt. $\angle$ s, $\overline{AD} \parallel \overline{BC}$ |
| $AC = AC$                           | common   |
| $\triangle ABC \cong \triangle CDA$ | ASA  |
| $\therefore AB = CD$ and $AD = BC$  | corr. sides of $\cong \Delta$ s.                         |

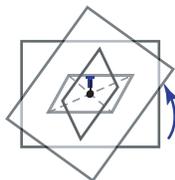


- $\therefore$  **Opposite sides are equal.**
- (2)  $\triangle ABC \cong \triangle CDA$
- |                                  |                                      |
|----------------------------------|--------------------------------------|
| $\therefore \angle B = \angle D$ | from part (1)                        |
| $\angle A = r + s$               | corr. $\angle$ s of $\cong \Delta$ s |
| $= x + y$                        | $\angle$ s add                       |
| $= \angle C$                     | $s = x, r = y$                       |
|                                  | $\angle$ s add.                      |

- $\therefore$  **Opposite angles are equal.**
- (3)  $\triangle ABC \cong \triangle CDA$
- |                                     |  |
|-------------------------------------|--|
| $r = y$                             | from part (1)  |
| $AD = CB$                           | corr. $\angle$ s of $\cong \Delta$ s                       |
| $b = d$                             | corr. sides of $\cong \Delta$ s                            |
| $\triangle ADP \cong \triangle CBP$ | alt. $\angle$ s, $\overline{AD} \parallel \overline{BC}$ . |
| $\therefore AP = PC$ and $BP = PD$  | ASA  |
|                                     | corr. sides of $\cong \Delta$ s.                           |



$\therefore$  **The diagonals bisect each other.**

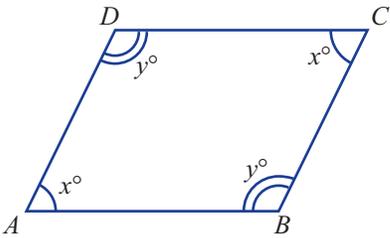
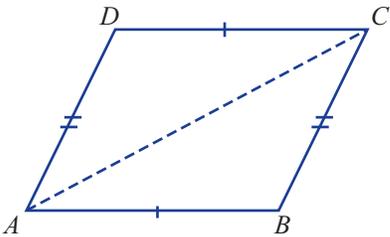
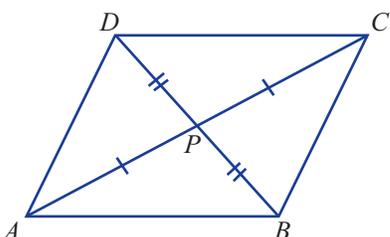
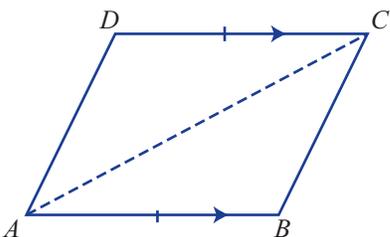


These three properties can also be seen by symmetry — but it is not a folding symmetry. Instead, put a pin at the center of parallelogram (the point where the diagonals intersect) and rotate  $180^\circ$  around that point. Do you see why this symmetry gives all three properties (opposite sides equal, opposite angles equal, diagonals bisect)?

**EXERCISE 4.1.** Try the activity suggested above using transparency sheet.

The converse of each of these three properties is also true: If a quadrilateral satisfies any one of the conditions (1), (2), (3) above, then that quadrilateral is a parallelogram. Students can be led to this conclusion by working through the following eighth-grade activity.

**EXERCISE 4.2.** The chart below gives four different criteria for recognizing parallelograms. Fill in the blanks to complete these sketches of proofs:

<p>If both pairs of opposite angles are equal, then the figure is a parallelogram.</p> 	$x + y + x + y = \underline{\hspace{2cm}} \quad \angle \text{ sum } n\text{-gon}$ $x + y = 180$ $\therefore \overline{AB} \parallel \overline{CD} \quad \text{int. } \angle\text{s converse.}$ $\therefore \overline{AD} \parallel \overline{BC} \quad \underline{\hspace{2cm}}$ $\therefore ABCD \text{ is a parallelogram.}$
<p>If both pairs of opposite sides are equal, then the figure is a parallelogram.</p> 	$\triangle ABC \cong \triangle CDA \quad \underline{\hspace{2cm}}$ $\angle BAC = \angle DCA \quad \underline{\hspace{2cm}}$ $\angle CAD = \angle ACB \quad \underline{\hspace{2cm}}$ $\therefore \overline{AB} \parallel \overline{CD} \quad \text{alt. } \angle\text{s converse}$ $\therefore \overline{AD} \parallel \overline{BC} \quad \text{alt. } \angle\text{s converse}$ $\therefore ABCD \text{ is a } \underline{\hspace{2cm}}$
<p>If the diagonals bisect each other, then the figure is a parallelogram.</p> 	$\triangle APB \cong \triangle CPD \quad \underline{\hspace{2cm}}$ $\angle PAB = \angle PCD \quad \underline{\hspace{2cm}}$ $\underline{\hspace{2cm}} \parallel \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}}$ <p>Similarly,</p> $\triangle APD \cong \triangle CPB \quad \underline{\hspace{2cm}}$ $\angle ADP = \angle CBP \quad \underline{\hspace{2cm}}$ $\underline{\hspace{2cm}} \parallel \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}}$ $\therefore ABCD \text{ is a } \underline{\hspace{2cm}}$
<p>If two opposite sides are equal and parallel, then the figure is a parallelogram.</p> 	$DC = AB \quad \text{given}$ $\angle DCA = \angle BAC \quad \text{alt. } \angle\text{s, } \overline{AB} \parallel \overline{DC}$ $AC = AC \quad \text{common}$ $\triangle ACD \cong \triangle CAB \quad \text{SAS}$ $\angle DAC = \underline{\hspace{2cm}} \quad \text{corr. } \angle\text{s, } \cong \Delta$ $\overline{AD} \parallel \overline{BC} \quad \underline{\hspace{2cm}}$ $\therefore ABCD \text{ is a } \underline{\hspace{2cm}}$

These criteria justify the “school definitions” of rhombus and rectangle (see page 46 and the comment on page 47). The standard definitions used in high school geometry are:

- A rhombus is a quadrilateral with all four sides equal in length.
- A rectangle is a quadrilateral with four right angles.

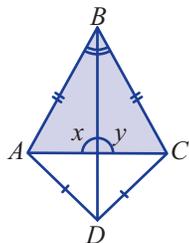
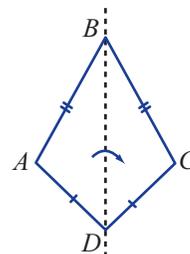
With these definitions, the second boxed criterion on the previous page shows that *every rhombus is a parallelogram* and the first criterion shows that *every rectangle is a parallelogram*. Thus we lose nothing by defining a rhombus to be “a parallelogram with four equal sides” and a rectangle to be “a parallelogram with four right angles”, as we did on page 46.

### Kites

By definition, a kite is a quadrilateral in which two adjacent sides have equal length and the remaining two sides also have equal length. A kite has the following properties:

1. There is a pair of equal opposite angles.
2. One of its diagonals bisects a pair of opposite angles.
3. The two diagonals are perpendicular.

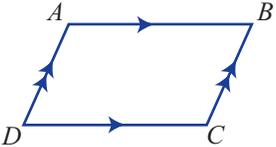
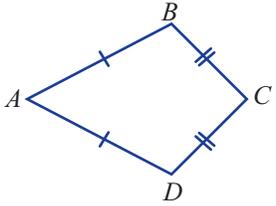
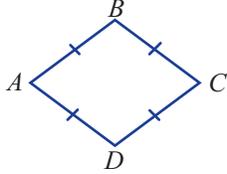
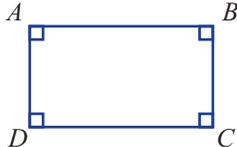
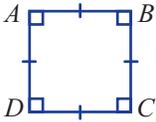
The first two of these properties are obvious from symmetry. In the kite on the right, the line of symmetry is  $\overline{BD}$ ; it splits the kite into two triangles that are congruent by SSS (or by symmetry). Therefore,  $\angle A = \angle C$ , which shows the first statement. Furthermore,  $\overline{BD}$  bisects  $\angle B$  and  $\angle D$  (by symmetry or because these are corresponding angles of congruent triangles), which shows the second statement.



To see that the diagonals are perpendicular, note that the two shaded triangles pictured on the left are congruent by SAS. Angles  $x$  and  $y$  are therefore equal, so both are  $90^\circ$ . Thus the diagonals are perpendicular.

### Quadrilateral Properties

We have seen that rectangles, rhombuses, and squares are also parallelograms. These figures therefore have all the properties of parallelograms. Likewise, rhombuses and squares are kites, so the properties of a kite also hold for rhombuses and squares. Below is a summary of the properties of quadrilaterals that are learned in grades K-8.

<p>By definition, a <i>parallelogram</i> is a quadrilateral in which both pairs of opposite sides are parallel. For parallelograms,</p> <ol style="list-style-type: none"> <li>1. both pairs of opposite sides are equal,</li> <li>2. both pairs of opposite angles are equal,</li> <li>3. the diagonals bisect each other, and</li> <li>4. two opposite sides are equal and parallel.</li> </ol> <p>Furthermore, any quadrilateral satisfying (1), (2), (3), or (4) is a parallelogram.</p>	
<p>By definition, a <i>kite</i> is a quadrilateral with two consecutive sides of equal length and the other two sides also of equal length. For kites,</p> <ol style="list-style-type: none"> <li>1. at least one pair of opposite angles are equal,</li> <li>2. there is a diagonal that bisects a pair of opposite angles, and</li> <li>3. the diagonals are perpendicular to each other.</li> </ol> <p>Furthermore, any quadrilateral satisfying (2) is a kite.</p>	
<p>By definition, a <i>rhombus</i> is a quadrilateral with all sides of equal length.</p> <p>A rhombus has all the properties of a parallelogram and of a kite, and the diagonals bisect the interior angles.</p>	
<p>By definition, a <i>rectangle</i> is a quadrilateral all of whose angles are right angles.</p> <p>A rectangle has all of the properties of a parallelogram, and its diagonals are equal.</p> <p>Furthermore, any quadrilateral whose diagonals are equal and bisect each other is a rectangle.</p>	
<p>By definition, a <i>square</i> is a rectangle with all sides of equal length.</p> <p>A square has the properties of a rectangle and of a rhombus.</p>	

### Curriculum Overview

Look how far we've come! In four chapters, we have followed the main thread of the geometry curriculum starting in Kindergarten and now we are reaching the level of beginning high school geometry. This upward path is built on two themes: solving word problems involving measurements, and solving unknown angle problems.

The part of the curriculum we have studied so far can be organized into four phases:

- **Grades 1–4.** Students learn to measure and calculate with lengths and angles.
- **Grades 4–6.** Students learn angle, triangle, and parallelogram facts (convincingly introduced by brief paper-folding, cutting, and measurement exercises). They use those facts to solve many unknown angle problems.
- **Grades 6–7.** Students learn parallel line facts and use them to give geometric proofs of some of the angle and triangle facts learned in grades 4–6. Students continue to solve many unknown angle problems, which now involve simple algebra.
- **Grades 7–8.** Congruence and similarity tests are introduced using constructions and measurement, and the Pythagorean Theorem is introduced with a simple proof. Students continue working on unknown angle problems and begin doing simple proofs.

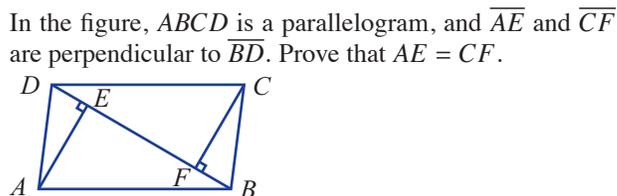
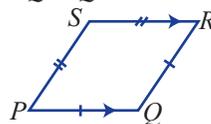
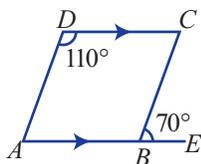
Students with this background are well-prepared for a 9th or 10th grade geometry course that concentrates on the logical structure of geometry. By that time they have spent five years developing knowledge and intuition for geometry and logical deduction. This built-up intuition and knowledge make learning the “axiomatic approach” to geometry in high school courses *enormously easier*.

We have reached the limits of elementary school mathematics. The next section briefly describes some aspects of the “transformation” approach to geometry which are included in some elementary school curricula. After that, we will return to second grade and study how area is developed in elementary school.

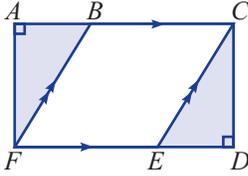
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### Homework Set 16

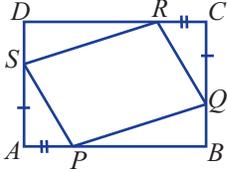
1. Do Problem 2 on page 282 of NEM1. Copy only the letters for the properties (you don't need to write the sentence). For example, the first line of the table should start: “(a) T F T ...”.
2. Do all parts of Problem 3 on pages 283–284 of NEM1. Do not give proofs or Teacher's Solutions — just find the unknowns  $x$  and  $y$  in each figure.
3. Prove that  $ABCD$  is a parallelogram.
4. In the figure,  $\overline{SR} \parallel \overline{PQ}$ ,  $PS = SR$ , and  $PQ = QR$ . Prove that  $PQRS$  is a rhombus, i.e., prove that  $PS = SR = PQ = QR$ .
5. In the figure,  $ABCD$  is a parallelogram, and  $\overline{AE}$  and  $\overline{CF}$  are perpendicular to  $\overline{BD}$ . Prove that  $AE = CF$ .



6. In the figure,  $ACDF$  is a rectangle and  $BCEF$  is a parallelogram. Prove that  $\triangle ABF \cong \triangle DEC$ .



7. In the figure,  $ABCD$  is a rectangle, and  $AP = CR$  and  $AS = CQ$ . Prove that  $PQRS$  is a parallelogram.

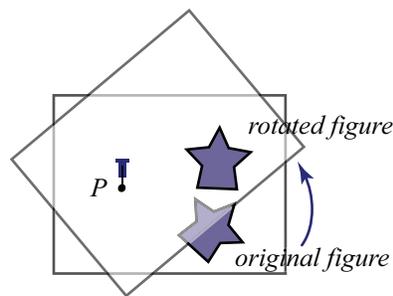


8. Give an Elementary Proof: the diagonals of a square are perpendicular. Start by drawing a picture and writing:  
 Given: Square  $ABCD$  with diagonals  $\overline{AC}$  and  $\overline{BD}$   
 To Prove:  $\overline{AC} \perp \overline{BD}$ .
9. Prove that a parallelogram with one right angle is a rectangle.

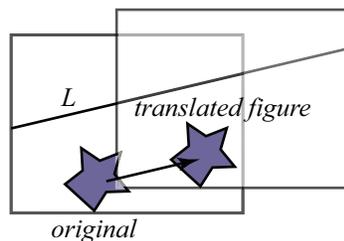
## 4.5 Transformations and Tessellations

This section describes two topics often used for “enrichment”: transformations (which sometimes appear under the title “motion geometry”) and tessellations (also called “tilings”). Both are serious subjects whose study is college-level mathematics. Both are also frequently taught in K-8 mathematics classes. At the K-8 level, these topics are rich in classroom activities, but the mathematical content sometimes gets lost. This section explains the mathematics that is involved and includes some additional “teacher-knowledge” information.

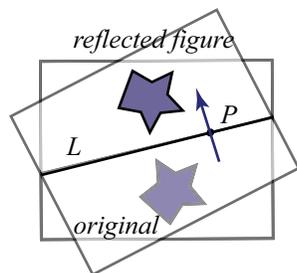
Here is one way to introduce the idea of transformations in the classroom. Begin by drawing a figure on a piece of cardboard. Lay an overhead projector transparency on it and trace the figure.



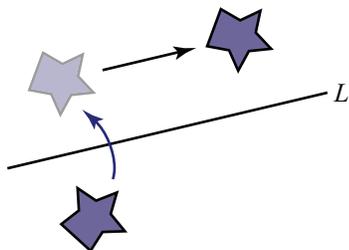
- (1) At some point  $P$ , insert a pin through the transparency into the cardboard (this is the reason for using cardboard). Rotate the transparency. The traced figure moves to a new position. This motion is called a *rotation about  $P$*  and the moved figure is called a *rotation* of the original figure. (In many elementary school illustrations, the rotation is around a point  $P$  that lies inside the figure and is often the center of the figure.)



- (2) Instead of inserting a pin, draw a line  $L$  on the cardboard and trace the line on the transparency. Slide the transparency along the line, keeping the traced line exactly on top of the line  $L$  while moving the figure to a new position. This motion is called a *translation parallel to  $L$*  and the moved figure is called a *translation* of the original figure.



(3) Now mark a point  $P$  on  $L$  and the corresponding point on the traced copy of  $L$ . Turn over the transparency and align the traced line exactly on top of  $L$  with the two marked points aligned. This produces a mirror image of the original figure. This motion is called a *reflection across  $L$* , and the moved figure is called a *reflection* of the original figure.



A *composite motion* is any succession of rotations, translations, and reflections. For example, reflecting across a line  $L$  and then translating in the direction of  $L$  is a composite motion called a *glide reflection*.

Transformations move the entire plane, including all figures within it. The transformed figure is called the *image* of the figure under the transformation. Notice that if you trace a segment or angle, moving the transparency does not affect the length or angle. Thus:

For rotations, translations, reflections and their composites:

- the image of a segment is a segment of equal length,
- the image of an angle is an angle of equal measure,
- the image of a figure is a congruent figure.

rigid  
motion

There is a more elementary version of these ideas that involves moving figures (rather than moving the entire plane). If students cut a figure out of cardboard and lay it on a sheet of paper, they are able to slide the figure around. The various ways of sliding and turning are called the *motions* of the figure. These are sometimes called *rigid motions* because the cardboard figure is rigid — it does not stretch or distort, and hence does not change lengths and angles.

In classroom language, a translation “moves the figure parallel to a line”, a rotation “spins the figure”, and a reflection “turns the figure over and places it on the other side of a given line”. These descriptions are not precise, but they are enough for students to recognize two key points:

- A motion takes a figure to a figure with the “same size and shape”.
- Every motion can be obtained as a succession of rotations, translations, and reflections.

These two points are the basis for the informal elementary school definition of congruence.

Two figures are *congruent* if one is obtained from the other by a rigid motion.

The required motion can be thought of as a way of picking up the first figure and laying it down

so that it exactly covers the second figure. Alternatively, one can think of sliding and flipping the first figure through a succession of rotations, translations, and reflections until it exactly matches the second figure.

Middle school students learn that congruence is defined by the requirement that corresponding sides and angles are equal. At that point a mathematical question arises: is it true that any two congruent figures are related by a succession of rotations, translations, and reflections? The answer is “Yes, that is a mathematical theorem,” but the proof is beyond K-8 mathematics. Nevertheless, the basic ideas, discussed next, are important teacher knowledge.

### Isometries

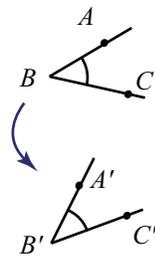
transformation

One unifying idea in geometry is the notion of an “isometry”. The mathematical study of isometries begins with the precise definition of a (general) *transformation* of the plane, which is defined in college-level mathematics as an “invertible mapping from the plane to the plane”. For our purposes, it is enough to say that a transformation is any succession of rotations, translations, reflections and any kind of stretching and pulling that might be possible with a transparency sheet made of stretchable plastic.

**DEFINITION 5.1.** *A transformation that takes every line segment to a line segment of equal length is called an isometry.*

Thus isometries are “length-preserving transformations”. They also preserve distances because the distance between two points is the length of a segment. The congruence tests for triangles show that isometries also preserve angles.

**THEOREM 5.2.** *Isometries preserve angles.*



Given:  $A'$ ,  $B'$ , and  $C'$  are obtained from  $A$ ,  $B$  and  $C$  by an isometry.

To prove:  $\angle B' = \angle B$

Constr.: Draw  $\overline{AC}$  and  $\overline{A'C'}$ .

*Proof.* 
$$\left. \begin{array}{l} A'B' = AB \\ B'C' = BC \\ A'C' = AC \end{array} \right\} \text{isometries preserve lengths}$$

$$\Delta A'B'C' \cong \Delta ABC \quad \text{SSS}$$

$$\therefore \angle B' = \angle B.$$

Theorem 5.2, together with Definition 5.1, shows that isometries preserve both lengths and angles. Consequently, they take triangles to congruent triangles. In fact, *isometries move any figure to one that is congruent to the original*.

identity

fixed points

Rotations, translations and reflections preserve lengths, so are isometries. What other kinds of isometries are there? The following theorem, which is proved in college geometry courses, provides a complete list. The statement involves two new terms. The isometry that does nothing — that does not move points, lines, and figures at all — is called the *identity*. In general, an isometry will move most points of the plane; the points that are not moved are called the *fixed points* of the isometry.

**THEOREM 5.3.** *All isometries are composites of rotations, translations and reflections. In fact, every isometry is one of four types: a rotation, a translation, a reflection or a glide reflection. Specifically,*

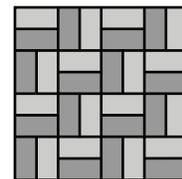
- a) *The only isometry that fixes 3 non-collinear points is the identity.*
- b) *Any isometry that fixes 2 points is a reflection across the line through those points or is the identity.*
- c) *Any isometry that fixes exactly 1 point  $P$  is either a rotation around  $P$ , or the composite of a rotation around a point  $P$  and a reflection across a line through  $P$ .*
- d) *Any isometry that fixes no points is either a translation or a glide reflection.*

To get a sense of a), put 3 pins through your tracing paper and try to move it. You can't! Inserting 2 pins, then 1 pin, will similarly provide intuition (but not a mathematical argument) for b) and c). Statement d) is less intuitive. It includes the fact, which is not at all evident, that the composite of a rotation and a glide reflection along a line  $L$  is a glide reflection along a different line.

The details are complicated. Understanding the reasoning requires mathematical knowledge and maturity beyond the K-8 level. In other countries, transformations are not introduced until after students have begun using coordinate systems in the plane. At that point, students have the geometric tools and experience to study transformations *as mathematics*. In contrast, many U.S. curricula and state standards introduce motions at the level of cutting and folding paper in elementary or middle school, and then abandon it. This is fine for enrichment and building intuition, but teachers should be aware that the study of transformations is not part of the main story-line of geometry in the lower grades, which focuses on measurement and deduction.

### Tessellations

One often sees floor tiles and wall decorations arranged in geometric patterns formed by joining together copies of geometric figures without gaps or overlaps. These patterns are called *tessellations* (the ancient Romans made mosaics from clay and glass tiles called *tessella*).

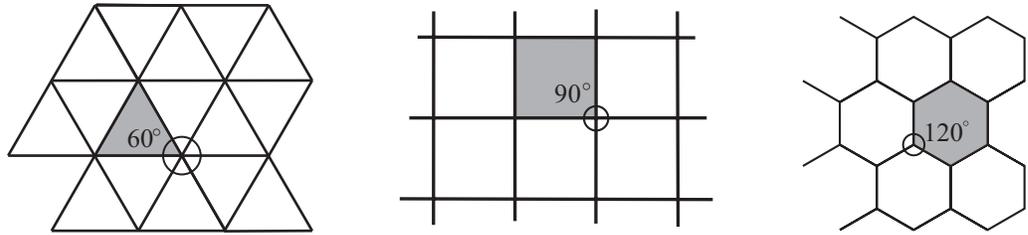


regular  
tessellation

More precisely, a *tessellation* or *tiling* of the plane is a collection of polygonal regions whose union is the entire plane and whose interiors do not intersect. A tessellation made of congruent regular polygons is called a *regular tessellation*.

It is a useful exercise to give students examples of tessellations and have them shade exactly one of the congruent polygons in the tiling. They can then visually test whether the tessellation is regular by checking the two requirements: a) is the tile a regular polygon? and b) are all the tiles in the tessellation congruent to the shaded one?

There are only three regular tessellations. The three possibilities are formed by equilateral triangles, squares, and regular hexagons as shown.



The proof that these are the only regular tessellations is an interesting classroom exercise in grades 5-7. The proof is based on the fact, which is evident from the pictures, that each vertex is surrounded by congruent angles. Thus, in a regular tessellation by  $n$ -gons

- (i) every interior angle of each tile is a factor of  $360^\circ$

and we know (from Section 3.3) that

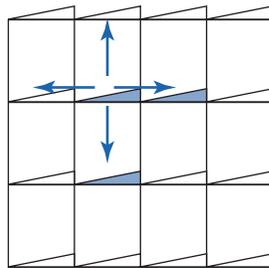
- (ii) interior angles have equal measure, and

- (iii) the sum of interior angles is  $180^\circ(n - 2)$ .

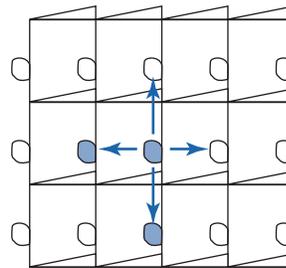
With these facts students can show, as you will in a homework problem, that the only regular tessellations are the three pictured above.

symmetry of a  
tessellation

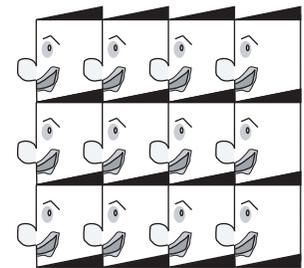
A *symmetry* of a tessellation is a transformation that moves the tessellation onto itself. For example, a tessellation by squares can be moved onto itself by translating one or more units up, down, left, or right, or by rotating  $90^\circ$  around the center point of one square. Fancy tessellations can be created by modifying simple tessellations by such transformations. Here is an example.



*Choose part of one rectangle and translate it to all rectangles.*



*Translate another part.*



*Paint your tiles for visual appeal.*

A *semi-regular tessellation* is a tiling using two or more tile shapes, each a regular polygon, so that the arrangement of polygons at every vertex point is identical (i.e., the same tiles surround each vertex in the same order). It is a mathematical theorem (too difficult for school mathematics) that there are exactly 11 such semi-regular tessellations (including the 3 regular tessellations). There are also non-periodic tessellations called “Penrose tilings”.

Elementary students can have fun with tessellations. Making and coloring tessellations is an excellent project for the end of the school year when summer is coming and students need a break from learning mathematics.

Homework Set 17

1. In this problem you will determine which regular  $n$ -gons tessellate. The steps outline a typical classroom exercise.
  - a) Recall that the sum of the interior angles of any  $n$ -gon is  $(n - 2)180^\circ$ . Use this fact and a sketch to explain why the measure of each interior angle for any  $n$ -gon is

$$I_n = 180^\circ - \frac{360^\circ}{n}.$$

- b) List all the factors of 360 greater than or equal to 60.
- c) Fill in the following table.

# of sides	Interior angle	Factor of 360°?	Tessellate?
3	60°	✓	✓
4			
5		✗	
6			
7			
8			
9			
10			

- d) Use the formula for  $I_n$  and some algebra to show that whenever  $n$  is bigger than 6,  $I_n$  is between  $120^\circ$  and  $180^\circ$ .
  - e) Explain why your answers to parts a)-d) show that regular  $n$ -gons tessellate only for  $n = 3, 4$  and  $6$ .
2. Study the last picture of this section. Follow the same procedure to make two simple tessellations. Draw at least 6 tiles for each. Coloring may make your picture clearer.
  3. Do the following constructions.
    - a) Draw a triangle  $\triangle ABC$  and a ray  $\overrightarrow{DE}$ . Show how to move the triangle  $\triangle ABC$  4 cm in the direction  $\overrightarrow{DE}$  using only a compass, ruler, and a set-square.
    - b) Draw a triangle  $\triangle ABC$  and a line  $l$ . Show how to reflect the triangle  $\triangle ABC$  across the line  $l$  using a compass and straightedge.
    - c) Draw a triangle  $\triangle ABC$  and mark a point  $D$ . Show how to rotate the triangle  $\triangle ABC$  about the point  $D$  an angle measure of  $45^\circ$ , using a compass and straight-edge.
  4. (*Study the textbook!*) Read pages 76–79 in Primary Math 5B. Use graph paper to show how Shape B on page 78 can tessellate (graph paper can be obtained from [www.printfreegraphpaper.com](http://www.printfreegraphpaper.com)).
  5. (*Study the textbook!*) Assuming that 1 to 2 pages can be covered in a day’s lesson, estimate how many class days fifth grade Primary Math teachers spend on tessellations.

6. Explain, with pictures, why *every parallelogram tessellates*.
7. Explain why *every triangle can tessellate* (this can be done with one sentence and one picture using your answer to Problem 6).
8. \* This problem takes you through a classroom explanation of the fact that *every quadrilateral tessellates*.

On a sheet of thin cardboard, draw a quadrilateral about 3 inches across. Make the sides straight (use a ruler) and make the four interior angles have clearly different measures (your quadrilateral needn’t be convex). Label the interior angles  $w, x, y, z$  in clockwise order.

- a) What is the sum  $w + x + y + z$  of the interior angles?

Cut out your quadrilateral with scissors; this is your “template”. On a large blank sheet of paper, make a copy of your quadrilateral by tracing around your template. Call this figure “Tile 1” and label its interior angles  $w, x, y, z$  as on the template. Draw additional tiles by repeatedly applying the following step (which is named to indicate the edge we fill across).

STEP  $xy$ : Without flipping the template, align the template along the edge between interior angles  $x$  and  $y$  of one of the previously-drawn tiles so that  $y$  on the template matches with  $x$  on the tile and vice versa. Trace around the template to create a new tile. Label the interior angles of the new tile  $w, x, y, z$  as on the template.

Do STEP  $xy$  to draw Tile 2. Then do STEP  $wx$  on Tile 1 to draw Tile 3.

- (b) Look at the vertex of Tile 1 labeled by angle  $x$ . It is surrounded by 3 tiled angles and an un-tiled angle. The un-tiled angle has the same measure as which interior angle of your template? Why?
- (c) Fill in the blanks: one can fill in this un-tiled angle either by doing STEP \_\_\_ on Tile \_\_\_ or by doing STEP \_\_\_ on Tile \_\_\_. Do these give the same result?
- (d) After drawing Tile 4 by your first answer to (c), explain why the paired sides of Tiles 2 and 4 have equal length.
- (e) Similarly, explain why the paired sides of Tiles 3 and 4 have equal length.
- (f) Draw at least 6 tiles. Can this tiling procedure be continued indefinitely?