

Geometric Figures

This chapter describes how elementary students are introduced to the world of geometry. As we have seen, children first learn to measure lengths and angles and to solve arithmetic problems with measurements. By the middle of elementary school they are prepared to start studying geometry as a subject in its own right.

Already by grade 2, geometry means much more than simply identifying figures and naming their parts. In grades 2–4, children learn about parallel and perpendicular lines and solve problems involving supplementary and vertical angles. Every year children do drawing activities using rulers and, later, protractors, set squares, and compasses. In grade 5 they begin solving problems involving lengths and angles within triangles and quadrilaterals.

The elementary geometry curriculum focuses on clear reasoning and simple geometric facts. There are no proofs; instead, facts are introduced using paper-folding or symmetry arguments. Reasoning skills are developed through daily problem sets — problem sets that can be great fun for both children and teachers.

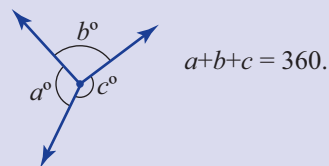
2.1 Fundamental Geometric Ideas

As children learn to measure in grades K-3 they acquire intuition about segments, angles and other objects of geometry. Sometime near the end of elementary school, intuition is replaced by precise definitions. This section describes how angles, perpendicular and parallel lines, and figures are presented intuitively, and then in the precise but child-friendly form of a “school definition”.

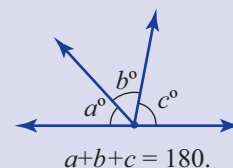
Angles

The “angle measurement facts” listed on page 24 have three especially useful consequences:

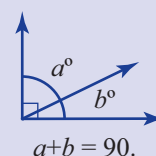
1. The total measure of adjacent angles around a point is 360° . (Abbreviation: \angle s at a pt.)



2. The total measure of adjacent angles forming a straight line is 180° . (Abbreviation: \angle s on a line.)



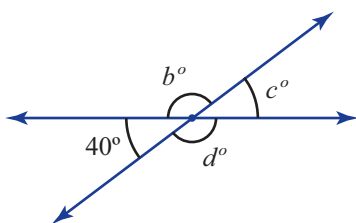
3. The sum of adjacent angles in a right angle is 90° . (Abbreviation: \angle s in rt. \angle .)



As you will see, these facts will be used repeatedly; they are the springboard that launches deductive geometry in grades 5 and 6. We have given each an abbreviation. Learning geometry is easier in classrooms where everyone consistently uses the same short, clear abbreviations.

When two lines intersect, they form four angles around the point of intersection. If we know the measure of any one of these, we can determine the measures of all four by recognizing pairs of supplementary angles.

EXAMPLE 1.1. In the figure, find the measures of angles b , c and d .



Solution: Moving around the vertex clockwise, we see successive pairs of supplementary angles:

$$40 + b = 180, \text{ so } b = 140.$$

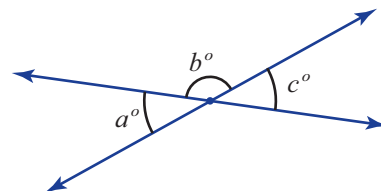
$$b + c = 140 + c = 180, \text{ so } c = 40.$$

$$c + d = 40 + d = 180, \text{ so } d = 140.$$

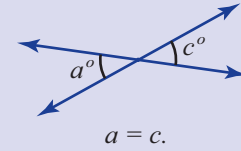
vertical
angles

In the figure below, angles a and c are a pair of *vertically opposite angles* because they are opposite each other through the vertex. For short, one says that a and c are *vertical angles*. Teachers should avoid the phrase “opposite angles” because pairs of angles that might be called “opposite” occur in other, different contexts (see page 48).

The figure contains two pairs of supplementary angles. Angles a and b are angles on a line, so a is $180 - b$. Similarly, c is also $180 - b$. Therefore $a = c$, so the two vertical angles have equal measure.



Vertical angles have equal measure.
(Abbreviation: **vert. \angle s.**)



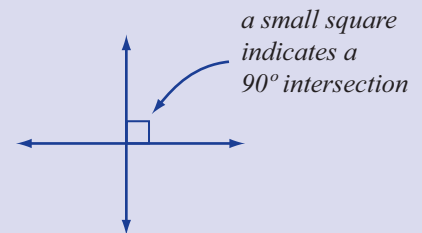
EXERCISE 1.2. Read pages 85–88 in *Primary Math 5A*. Which angle facts are introduced on these pages?

Perpendicular and Parallel Lines

As we saw in Section 1.4, students are introduced to right angles in *Primary Math 3B*. Then in *Primary Math 4A*, they learn that a right angle measures 90° and, a few pages later, they learn the term “perpendicular”.

DEFINITION 1.3. Two segments, rays, or lines are **perpendicular** if the lines containing them intersect to form a 90° angle.

If \overleftrightarrow{AB} is perpendicular to \overleftrightarrow{CD} , we write $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

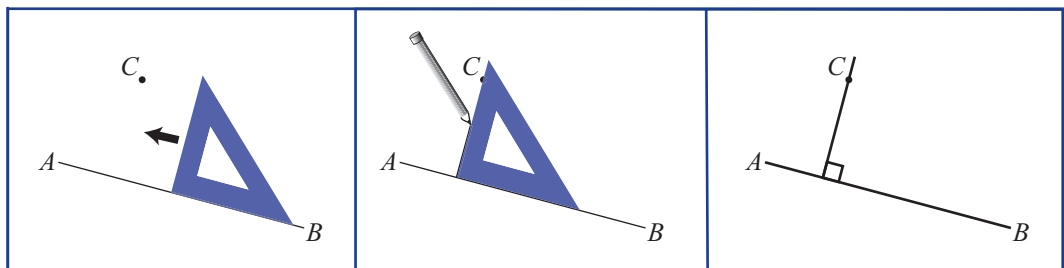


Two intersecting lines form four angles. If one of those angles is 90° , then by symmetry each of the other angles must be 90° .

Drawing perpendicular lines is harder than recognizing them because it requires motor skills. Hands-on activities drawing perpendicular lines deepen students’ understanding and develops skills that will be useful later. In classrooms, right angles are usually drawn with the aid of a *set square* (or *plastic triangle*), although any object with a right angle such as a piece of cardboard can be used.

set square

EXAMPLE 1.4. Use a set square to draw a perpendicular to the line \overleftrightarrow{AB} through the point C .



parallel lines

Two lines are *parallel* if they lie in the same plane and do not intersect. In addition, in K-8 geometry — but not always in high school geometry — a single line is considered parallel to itself. When lines \overleftrightarrow{AB} and \overleftrightarrow{CD} are parallel, we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. In pictures, pairs of matching arrows are used to indicate that two lines are parallel.



parallel segments

Two segments are parallel if they are part of parallel lines.

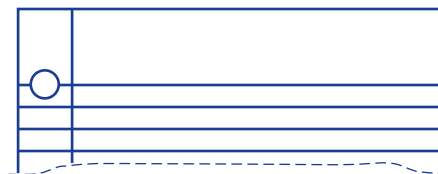
As always, the word “line” means a straight line that extends indefinitely in both directions. Consequently, to make sense of the phrase “do not intersect” children must envision extending the segments on their paper indefinitely into space, past distant galaxies — something that is not very concrete. Furthermore, the condition that two lines do not intersect does not suggest a way of drawing parallel lines, or a way to determine when two segments are parallel. Consequently, a different definition of parallel lines is used in elementary school.

DEFINITION 1.5 (School Definition). *Two lines, segments, or rays are **parallel** if they lie in the same plane and are both perpendicular to a third line.*

This school definition gives a mental picture that is concrete and easily explained. Examples of segments that have a common perpendicular are all around us, while there are few examples of non-intersecting lines extending indefinitely into space past distant galaxies.

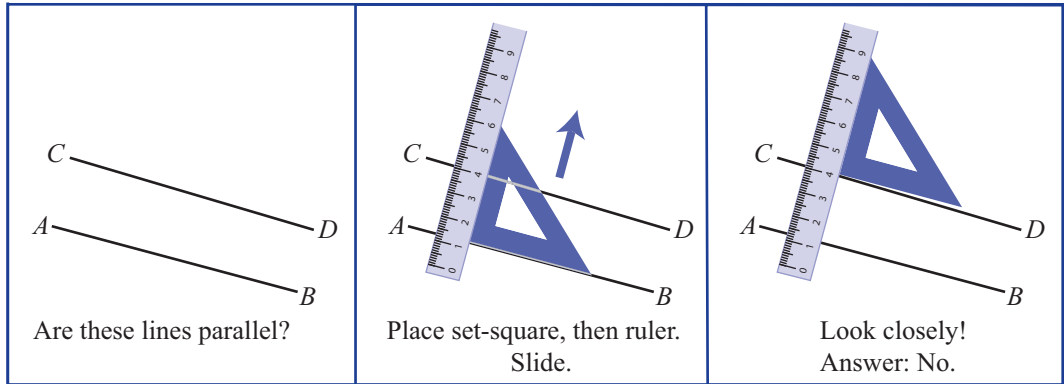


Railroad tracks and lined paper have many parallel lines.

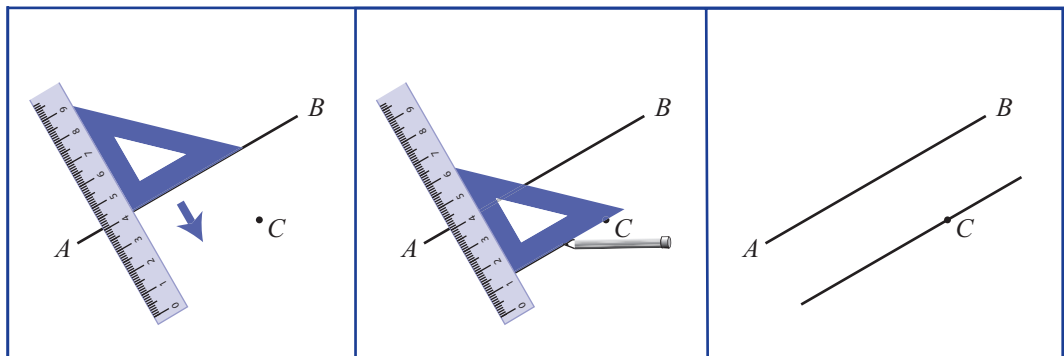


After describing parallel lines, the Primary Mathematics curriculum focuses on two activities to help students develop intuition about parallel lines: a method for determining whether two lines are parallel and a method for constructing parallel lines using a set square. The first method calls children’s attention to a fact that is treated as common knowledge in elementary mathematics: if two lines are both parallel to a third line, then they are parallel to each other.

EXAMPLE 1.6. Use a set square to determine whether two lines are parallel.



EXAMPLE 1.7. Use a set square and a ruler to draw a line parallel to \overleftrightarrow{AB} through the point C .

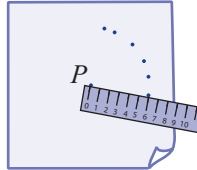
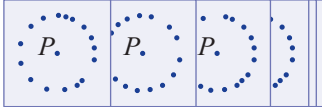
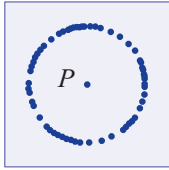


The Primary Math books include a marvelous set of problems involving parallel lines within parallelograms and other figures (see Section 2.4). Then in seventh grade, students study perpendicular and parallel lines more abstractly.

Circles

Children learn to recognize and name circles in kindergarten and first grade, but it is usually not until third or fourth grade that they encounter the precise definition of a circle. At this time they learn the terms “radius” and “diameter”, and learn to draw circles with compasses. Here is a three-step teaching sequence for introducing circles:

Step 1 — Definition. Circles are defined in terms of distance, not visual shape. The following activity helps children understand the key idea.

 <p>Mark point P on a transparency. Then mark as many points as you can that are 6 cm from point P.</p>	 <p>Put your transparency on top of your classmates' papers.</p>	 <p>What do you notice?</p>
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This activity shows that (i) a circle is a collection of points, and (ii) a circle is completely determined by its center and its radius. With these two realizations, children understand the essence of the mathematical definition

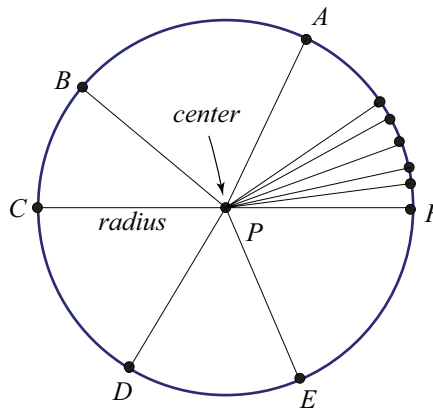
circle
center
radius

DEFINITION 1.8. Choose a point P in the plane and a distance R . The **circle** with center P and radius R is the set of all points in the plane that are distance R from the point P .

radius

The word “radius” has two meanings. *The radius* of a circle is a distance, as in Definition 1.8, while *a radius* is any segment with one endpoint on the center and the other endpoint on the circle. (The plural of this word is “radii”, pronounced “ray-de-eye”.) Because double meanings engender confusion, alert teachers always clarify whether “radius” refers to a distance or a segment.

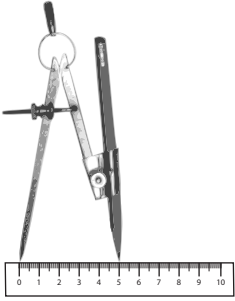
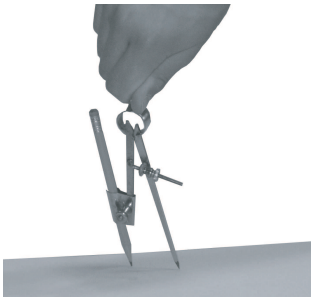
In a circle *all radii have the same length*. Teachers can reinforce this aspect of the definition by having students use rulers to check that all radii have the same length.



compass

Step 2 — Drawing Circles. A *compass* is a tool for drawing circles. Learning to use a compass is a prerequisite for later geometry.

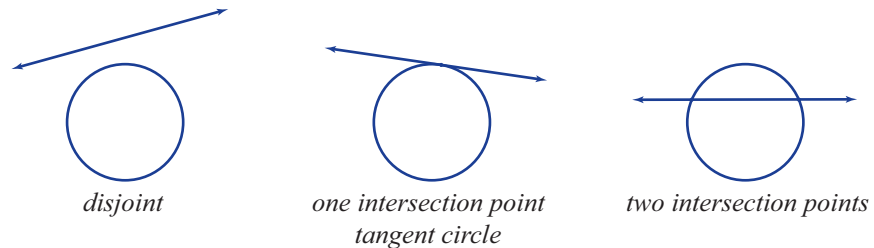
EXERCISE 1.9. Mark a point P on your paper. Draw a circle with center P and radius 5 cm.

 <p>Adjust your compass to 5 cm.</p>	 <p>Place the compass point on P. Rotate to draw a circle.</p>	<p>Use one hand, not two.</p> <p>Lean the compass forward and pull - don't push - the pencil.</p> <p>Don't let the point of the compass slip off P.</p>
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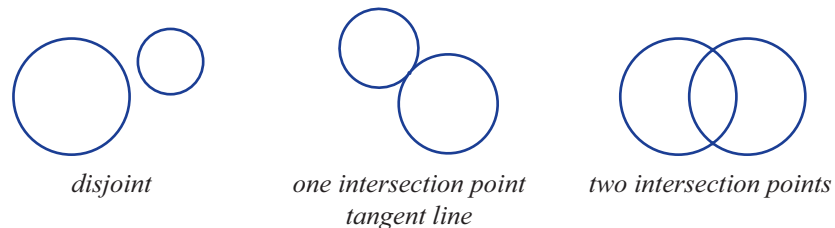
As in Definition 1.8, a circle is determined by choosing a center point and choosing a radius. These are exactly the choices students make when they use a compass to draw circles. Other methods for drawing circles, such as tracing around a tin can, hide those choices. The simplicity of the definition gets lost!

Using a compass to draw circles is easy, but requires some practice. The quality of the compass is a factor here. Compasses with screw adjustments work well, but the arms of low-quality compasses tend to slip, making it frustratingly difficult to maintain a constant radius while drawing circles.

Step 3 — Properties of Circles. A line and a circle can intersect in zero, one, or two points.



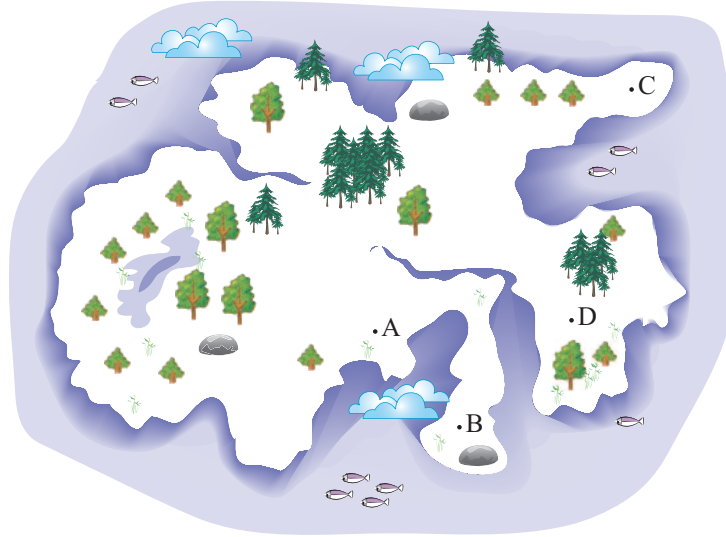
Likewise, two different circles intersect in zero, one, or two points.



EXERCISE 1.10. Use a compass and a straightedge to draw two circles that are tangent. (Hint: Draw a circle with center P and a ray with endpoint P . How can you choose a center point and a radius for a second, tangent circle?)

EXERCISE 1.11. *The following third grade problem describes circles that intersect in two points, but the problem has only one solution. Why?*

Gold is buried 4 cm from point B and 2 cm from point C. Use your compass to find the treasure.



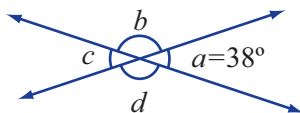
The teaching sequence continues in Primary Mathematics textbooks in sixth grade with definitions and applications of terms like diameter, circumference, and semicircle (see pages 26–29 of Primary Math 6B). We will examine this phase of the curriculum in Chapter 8.

Homework Set 5

- For each of the following times of day, sketch a clock face showing that time and find the measure (in degrees) of the angle formed by the hour hand and the minute hand. (Remember that the hour hand moves 30° per hour.)

a) 3:00 b) 3:30 c) 10:30 d) 2:45.

- In the figure, $a = 38^\circ$. Find b , c , and d .

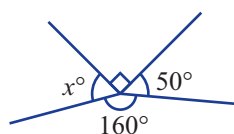


- On page 238 of NEM1, do Problem 1 of Class Activity 2.
- (Study the textbook!) Read the rest of Class Activity 2. These exercises have students accept three basic facts after checking them in one example. These are not proofs; the purpose of the activity is to make clear *what the facts*

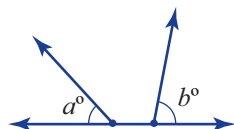
mean and help make them evident to the students (rather than being teacher-announced truths).

- Copy down the three facts with their abbreviations. We will shorten the first two abbreviations to “ \angle s on a line” and “vert. \angle s”.
 - Are these the same three facts introduced on pages 85–86 of Primary Math 5A?
- In Exercise 9.2 on pages 240–241 of NEM1, do problems 1b and parts b), f), h) and j) of Problem 2. Write your solutions in the manner of Worked Examples 1 and 2 on pages 239 and 240 on NEM 1. Be sure to include reasons in parentheses.
 - (Study the textbook!)
 - Read pages 78–83 of Primary Math 4A. Does the book define the terms perpendicular and parallel, or does it just show examples?
 - Now read pages 242–243 of NEM1. Does NEM1 define the terms perpendicular and parallel?

- c) Give a one or two sentence explanation of how Problem 2 on page 80 of Primary Math 4A helps students make sense of the seventh grade definition of perpendicular.
- d) Give a similar explanation of how Problem 2 on page 83 helps students make sense of the seventh grade definition of parallel lines.
7. On page 243 of NEM1, do Problem 1 of Class Activity 3 using a ruler and a set square. Then write a precise definition for the term *perpendicular bisector*.
8. (*Common Student Error*) When asked to find x in the figure below, Mary writes “ $x = 50$ ” and explains “because vertical angles are equal”. What is x actually? What erroneous assumption did Mary make?



9. (*Common Student Error*) In the figure, Jerry claims that “ $a + b = 180$ ” stating as the reason, “ \angle s on a line.”



There may be several reasons why Jerry is making this error. He may not understand angle measurements or how

to add angle measurements, he may not understand that to apply this fact the angles must be _____, or he may not understand that there are _____ degrees on a straight angle.

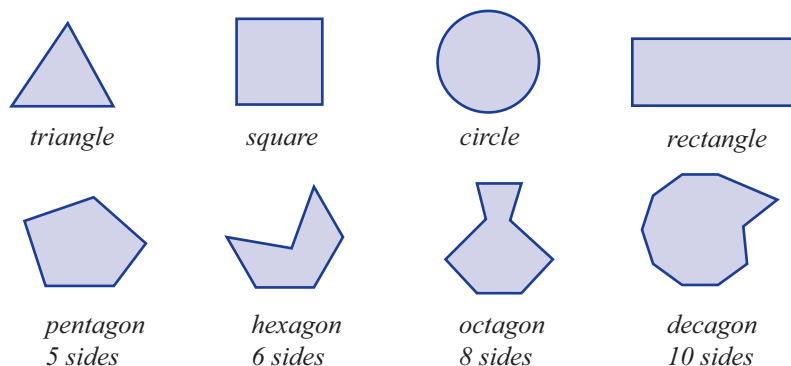
10. Draw a line L and a point P not on L . Using a ruler and set square, follow the procedure of Example 1.7 in this section to draw a line through P parallel to L .
11. A teacher asks her students for a precise definition of the term *circle*.
- Sarah says “A circle is a round segment with no endpoints”. Name at least two things wrong with Sarah’s definition.
 - Michael says “A circle is 360° ”. What two notions is Michael confusing?
 - Write down a precise definition of the term *circle*.
12. Practice drawing circles with a compass until you can draw complete circles without stopping. Then on your homework paper draw circles of radius 2 cm, 6 cm and 10 cm, using your ruler to set the compass width.
13. Use your compass to draw a circle; label your circle C and its center O . Choose any point on the circle and label it P . Draw another circle of radius OP with center P .
14. Use your compass to draw circle B ; label its center O . Choose any point on the circle and label it Q . Draw two more circles of different radii such that both circles intersect circle B only at point Q . (Hint: Draw line \overleftrightarrow{OQ} first.)

2.2 Triangles

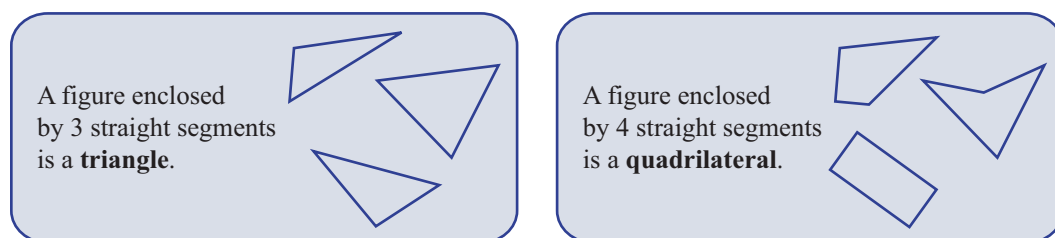
This section explains how an elementary school teacher might introduce one of the most important ideas in geometry: the fact that the sum of measures of the interior angles in any triangle equals 180° . To set the stage, we briefly discuss how children are introduced to shapes and figures in grades K-2.

Geometric Figures in the Early Grades

Pre-school, kindergarten, and first grade children learn to name, identify and draw 10 – 15 types of figures in the plane. There are two distinct levels to this learning. At the lower level, students identify shapes by sight-recognition: a square is a shape that *looks* like a square. Shape-naming is easy because young children have an innate ability to recognize and match shapes and are dazzlingly quick at learning new vocabulary words. At the higher level, students learn a precise definition and learn to check that the figures satisfy the required conditions. A “hexagon”, for example, is not simply something that looks like a hexagon. Rather, a hexagon is a figure with six straight sides — count ‘em to be sure! The teacher’s role is to move students from the first level to the second.

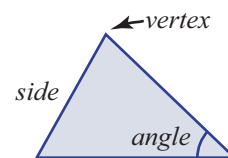


Learning to apply precise definitions is not automatic or easy. Children do not naturally think in precise terms, so they need guidance from their teacher. School textbooks facilitate learning by presenting **school definitions**: definitions that are precise and understandable to young children, but may not be the complete mathematical definition. In the previous section we saw how a school definition is used to introduce parallel lines. Here are two other common school definitions:



Grade 2 School Definitions

School definitions evolve and the wording becomes more sophisticated in later grades. But at the start, school definitions like those above help focus children's attention on the *parts* of the figure. The parts are given names: vertices, sides and angles. The region enclosed is called the *interior* of the figure; the interior does *not* include the sides or the vertices.



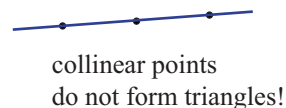
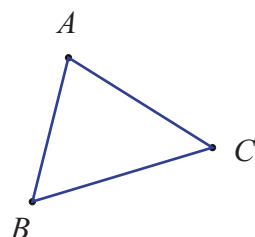
Figures are classified by their parts, not their appearance. For example, to distinguish an octagon (“a figure enclosed by 8 segments”) from a decagon (“... 10 segments”) one must *know the definition* and one must *count the number of sides*. Visual recognition is not enough. With practice, young children quickly become proficient at this.

The mathematical definition of triangle, typically given sometime in grades 4-7, is not just more sophisticated than the second grade definition, it is actually different.

DEFINITION 2.1. A *triangle* consists of three non-collinear points and the line segments joining them. (The points are called the vertices, and the segments are called the sides of the triangle.)

Notice what has changed: in grades K-2, a triangle included the vertices, sides, *and the interior*. Definition 2.1 switches children’s thinking to the more abstract “adult” meaning, in which a triangle consists of vertices and sides, but not the interior.

A triangle with vertices A , B and C is denoted $\triangle ABC$. Because vertices are not collinear, every triangle has an interior and an exterior.

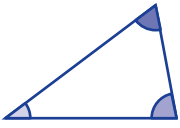

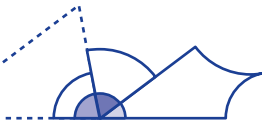


The Sum of the Angles of a Triangle

Primary Math 5B introduces students to some initial facts about triangles, squares, and other quadrilaterals. Almost immediately, students begin using these facts to solve multi-step geometry problems. There is a pedagogical problem here: students are comfortable using “facts” only after they are certain of their correctness, yet 5th graders lack the background to understand geometric proofs. How is this paradox avoided?

The Primary Math books have an ingenious solution called *picture proofs*. A picture proof is an illustration or activity, often involving folding or cutting-and-rearranging, that makes some fact evident. It both *clarifies the meaning of the fact* and *convinces students that it is true*. A good picture proof conveys the essential ideas of a mathematical proof without words.

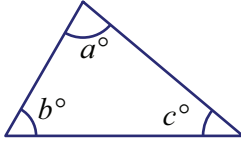
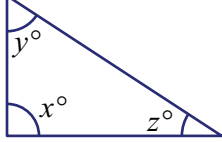
Many elementary textbooks demonstrate that the sum of angles in a triangle is 180° using one or both of the following activities. The first is a picture proof.

 <p>Make a paper triangle. Color its angles.</p>	 <p>Cut the triangle into three parts as shown.</p>	 <p>Put the three corners together.</p>	<p>Do the three angles form a straight line?</p> <p>What is the sum of the three angles of the triangle?</p>
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This activity gives compelling evidence that the sum of the three cut-out angles is 180° . But the triangle is destroyed in this process, so it is not clear that the fact is a result about triangles. Consequently, textbooks often provide a second activity in which students measure and add the angles of several triangles.

Measure and add the angles.

What did you notice?

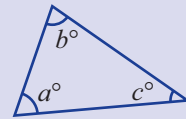
	
$a =$ _____ $b =$ _____ $c =$ _____ $a + b + c =$ _____	$x =$ _____ $y =$ _____ $z =$ _____ $x + y + z =$ _____

EXERCISE 2.2. Read pages 57-58 of *Primary Mathematics 5B*. Which exercise(s) on these pages is similar to the first activity? To the second activity? Roughly how much class time do you think is spent on doing the activities shown on pages 57-58?

The purpose of these activities is *not* to provide experimental evidence about the sum of the angles — geometry is not an experimental subject! Rather, these activities are designed to ensure that all students completely understand the meaning of the following statement.

In any triangle, the sum of the measures of the interior angles is 180° .

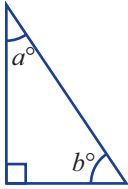
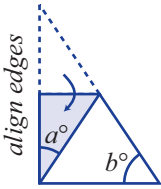
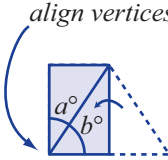
(Abbreviation: \angle **sum of Δ** .)



$$a + b + c = 180.$$

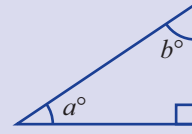
This is one of the most important facts in all of geometry. Almost all subsequent work in geometry depends upon it.

Here is a different teaching sequence leading to the same fact. This approach starts by studying *right triangles*. Students begin by doing the following folding exercise.

			<p>What do you notice?</p> $a + b = 90.$
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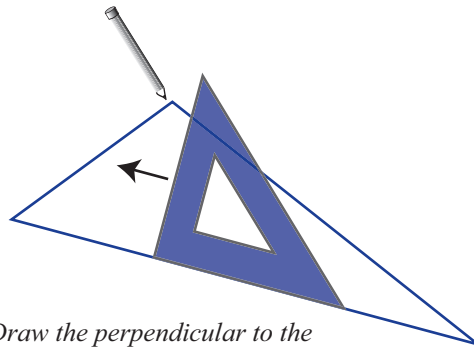
When one angle of a triangle is a right angle, the measures of the other two angles add up to 90° .

(Abbreviation: \angle **sum of rt. Δ** .)

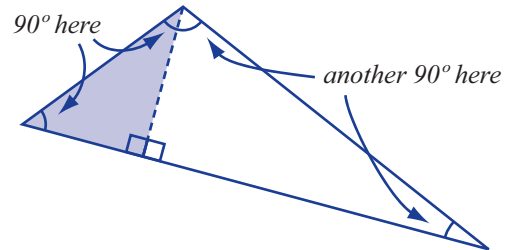


$$a + b = 90.$$

Students can then use this fact to find the sum of the angles of any triangle. The trick is to realize that every triangle can be split into two right triangles. The pictures below show how to use a set square to draw the splitting. *Be sure that you understand the reasoning depicted in the right-hand picture.*



Draw the perpendicular to the longest side through the opposite vertex.

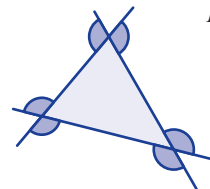
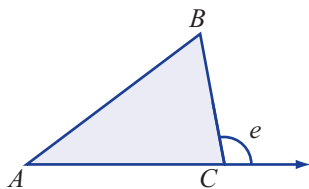


The angle sum is 180° .

Many geometric facts, like this one, become clear if one first thinks about right triangles, and then decomposes general figures into right triangles. The idea of decomposing figures into right triangles is an important recurring theme in geometry; we will use it repeatedly in Chapters 5 through 8. Teachers can add coherence to classroom discussions by emphasizing the role of right triangles in geometry.

Exterior Angles

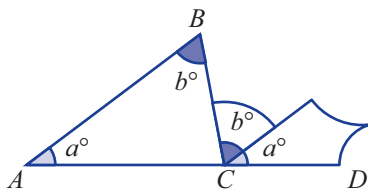
An angle formed by one side of a triangle and the straight extension of another side is called an **exterior angle** of the triangle. In the picture below, one side of triangle ABC has been extended, creating an exterior angle e . This exterior angle e is supplementary to the interior angle C .



Every triangle has 3 pairs of exterior angles.

At each vertex, interior and exterior angles are supplementary.

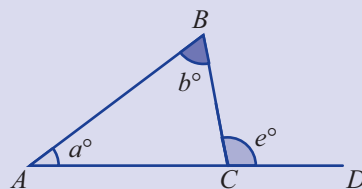
Now look again at the cut-out triangle on page 37. What do you notice?



This is a picture proof that the measure of the exterior angle $\angle BCD$ is $a + b$ degrees, which is the sum the two interior angles along the opposite side of the triangle.

The measure of each exterior angle of a triangle is the sum of the measures of the opposite interior angles.

(Abbreviation: **ext. \angle of Δ .**)



$$e = a + b.$$

Homework Set 6

In this homework set you will study two presentations of the facts about the angle sum of a triangle: one for 5th graders and one for 7th graders.

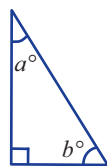
1. (Study the Textbook!) The fact that the sum of the measures of the angles of a triangle is 180° is introduced in pages 57-60 of Primary Math 5B.

- Do the experiment described on page 57. Is this a “picture proof” as defined in this section?
- The statement (at bottom of page 57) abbreviates language in the interest of clarity. Use the words “measure” and “interior” to expand this statement to make it complete and correct.
- Using a protractor, measure the angles of triangles A and B in Problem 1 on page 58. What do you get for the sum of the interior angles?
- What do students gain by doing this problem? (Hint: read this section!)

2. (Study the Textbook!) Continue reading and doing the problems in Primary Math 5B pages 58-60. Notice that the students are no longer measuring angles. Instead they are *figuring out angles using the given information*.

- Answer Problems 2 and 3 on page 58.
- Do the folding experiment described on page 59. Again, according to this section, what is the purpose of such picture proofs?

- Answer Problems 5 and 6. Write the answers as a list of numbers.
- Here is another 5th grade explanation of the boxed statement on page 59. Justify the first step by filling in the blank using the abbreviation for one of the boxed facts in this section



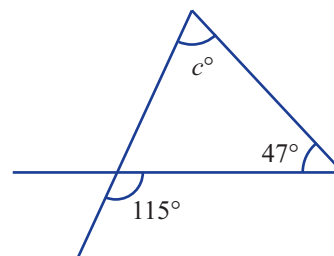
The figure shows a right triangle.

To Prove: $a + b = 90^\circ$.

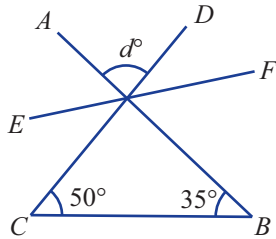
$$a + b + 90 = 180 \quad \underline{\hspace{2cm}}$$

Therefore $a + b = 90$.

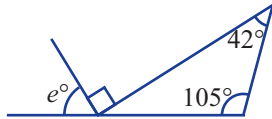
3. Find the value of c in the figure.



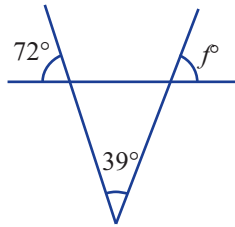
4. In the figure, \overline{AB} , \overline{CD} and \overline{EF} are segments. Find the value of d .



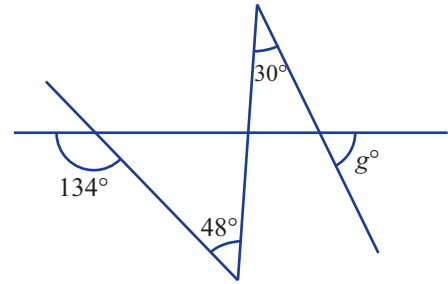
5. Find the value of e in the figure.



6. Find the value of f in the figure.



7. Find the value of g in the figure.



8. (*Study the textbook!*) In NEM, read pages 262-263 and study the solution to Worked Example 4 on page 263. Then in Exercise 10.2 (pages 263-266):
- Write solutions to Problems 3a, 3c, 3h, and 3i using the format of Worked Example 4. Be sure to include reasons in parentheses.
 - Use the same format to write solutions to Problems 4a, 4h, and 4l.

2.3 Symmetry and Triangles

A major portion of geometry is the study of the logical relationships between lengths and angles in figures. It is an excellent avenue for teaching deductive reasoning: children learn reasoning skills as they solve geometry problems. In the homework of the previous section, we saw how three facts about triangles open up a wealth of interesting, instructive geometry problems. This section adds a new fact to the repertoire: base angles of isosceles triangles are equal.

There are several routes to introducing isosceles triangles in the classroom. One common approach uses symmetry and paper-folding exercises to avoid formalities that are inappropriate at the K-8 level. As a result, teachers can quickly present the key facts about isosceles triangles and immediately engage students in solving interesting geometry problems.

Lines of Symmetry

Our visual system enables even young children to match shapes and recognize symmetry. This ability can be used by teachers to introduce geometric concepts and provide intuitive justifications. For this reason, most elementary curricula include a discussion of symmetry.

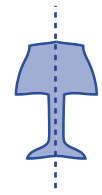
Symmetry is taught through examples. The immediate goal is to introduce words for two types of symmetry. These words help make children aware of visual patterns that they intuitively recognize. They also provide vocabulary for classroom discussions.

line of symmetry

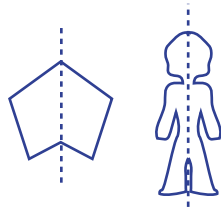
The idea of a *line of symmetry* can be explained by folding and cutting as shown. It is apparent that many familiar shapes have a line of symmetry. The pictures below show several facts about lines of symmetry that students should know.



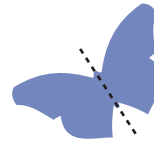
Cut folded paper along a path that begins and ends on the fold.



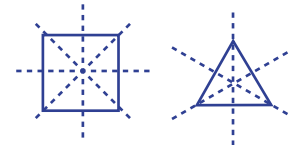
Unfolding creates a figure with a line of symmetry along the fold.



A line of symmetry separates the figure into two parts of equal size.



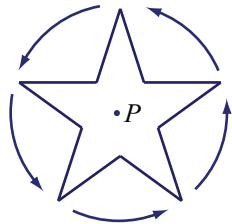
Lines of symmetry needn't be vertical.



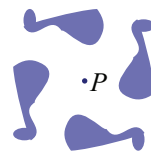
Some figures have several lines of symmetry.

rotational symmetry

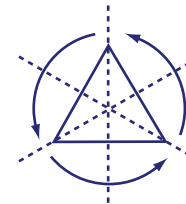
A second type of symmetry is rotational symmetry. A figure has *rotational symmetry of order n* if there is some “center” point P so that a rotation with center P by $1/n$ of a turn carries the figure onto itself. This can be illustrated by overlaying two copies of a figure, one on cardboard and one on a transparency, inserting a pin at the center point, and rotating.



Rotational symmetry of order 5.



The center of symmetry needn't be part of the figure.

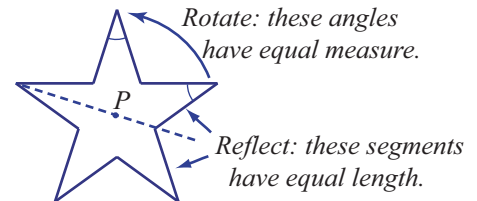


Many figures have both rotational symmetry and lines of symmetry.

Symmetry is an important but relatively easy idea. It can be introduced in any grade and needs only a few class sessions to teach. The Primary Mathematics curriculum includes two days on lines of symmetry in grade 4 and a week on symmetry in grade 7 (see Chapter 11 of NEM 1). The goal is to attune students to symmetry and to begin using symmetry to relate the measures of segments and angles in a figure.

When a figure has symmetry

- corresponding segments have equal length, and
- corresponding angles have equal measure.

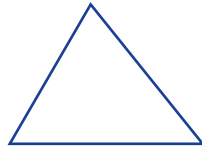


Teachers can use symmetry to clarify geometric reasoning. For example, symmetry helps students spot congruent triangles (such as the two “halves” of an isosceles triangle). Symmetry is also useful when computing areas: to find the area of a symmetric figure, one can find the area

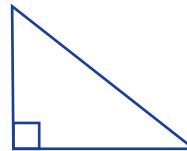
of half the figure, and double. As you continue reading this and later chapters, pay attention to the ways that you can use symmetry in the classroom.

Types of Triangles

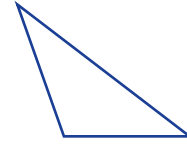
Triangles can be classified by their angles:



Acute Triangle
all angles are
less than 90° .

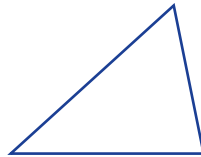


Right Triangle
one angle is 90° .



Obtuse Triangle
one angle is
greater than 90° .

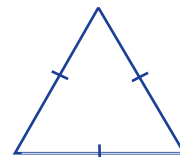
Triangles can also be classified by their sides, or by their lines of symmetry:



Scalene
all sides are unequal
no lines of symmetry



Isosceles
at least two sides equal
at least 1 line of symmetry



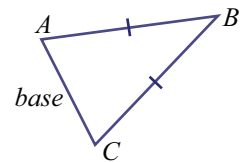
Equilateral
all sides are equal
3 lines of symmetry



EXERCISE 3.1. Is every equilateral triangle an isosceles triangle?

In an isosceles triangle, the sides of equal length are called the *legs* and the third side is called the *base*. The angles along the base are the *base angles* and the angle opposite the base is the *vertex angle*. Notice — and make sure that your students notice — that the base of an isosceles triangle need not be horizontal!

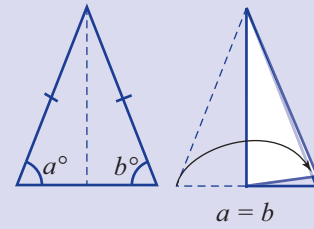
The base need
not be horizontal!



Isosceles and Equilateral Triangles

The definitions of an isosceles and equilateral triangles refer to side lengths, but say nothing about angles. But the definition determines a symmetry, which can be used to relate angles. An isosceles triangle has a line of symmetry, namely the line from the middle of the base to the opposite vertex. When an isosceles triangle is folded along its line of symmetry, the two sides and the two base angles match. Hence *the base angles are equal*.

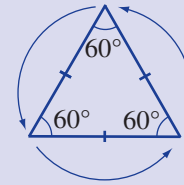
Base angles of an isosceles triangle are equal.
 (Abbreviation: **base \angle s of isos. Δ .**)



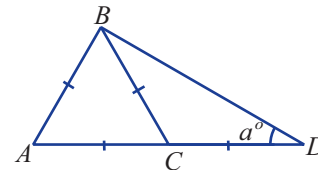
The same folding picture shows a second, logically different fact about isosceles triangles: If two angles in a triangle are equal then the opposite sides have the same length.

Equilateral triangles have rotational symmetry. If we rotate by $1/3$ of a turn, the triangle matches itself. Thus all three angles are equal. Since their sum is 180° , each is a 60° angle.

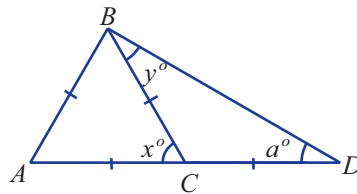
Each angle in an equilateral triangle is 60° .
 (Abbreviation: **equilat. Δ .**)



EXAMPLE 3.2. In the figure, $AB = AC = BC = CD$. Find the value of a .



Solution:



Mark angles x and y as shown.

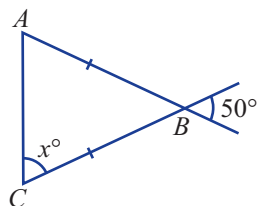
$$\begin{aligned}
 y + a &= x && \text{ext. } \angle \text{ of } \Delta \\
 x &= 60 && \text{equilat. } \Delta \\
 y &= a && \text{base } \angle \text{s of isos. } \Delta \\
 \therefore 2a &= 60, \\
 \therefore a &= 30.
 \end{aligned}$$

The symbol \therefore that appears above is a standard mathematical abbreviation for the word “therefore”.

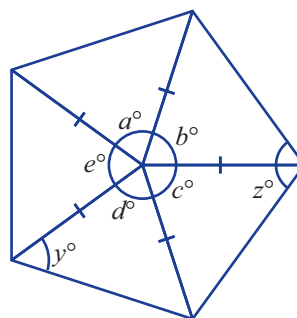
Homework Set 7

In this homework set you will study the presentation of the facts about angles of a triangle, first at the grade 5 level, then at the grade 7 level.

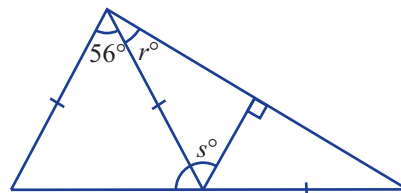
- (Study the Textbook!) The basic facts about equilateral and isosceles triangles are introduced in Primary Math 5B. Read pages 61–64 carefully and answer the following questions.
 - The fact that the base angles of an isosceles triangle are congruent is illustrated how?
 - Read Problem 2 on page 62 carefully. This problem illustrates the fact that “if a triangle has two equal angles then _____”.
 - Write the answers to Problems 3 and 4.
 - Problems 5, 6 and 7 are 2-step problems that combine facts about isosceles triangles with one other fact. For each, write the answer and state the “other fact” used.
 - Write the answers to Problem 8.
- Find the value of x in the figure.



- The figure shows a pentagon whose central angles are all equal: $a = b = c = d = e$. Find the value of y and the value of z .



- Find the value of r and of s in the figure.



- (Study the textbook!) Read Section 10.3 in NEM1, pages 268–270. Do Problems 1, 2a, 2b, 3a, 3b, 3c, 3d, and 3g.
- (Study the textbook!) Read Sections 11.1 and 11.2 in NEM1 pages 290–298. Do problem 4 on page 297 by drawing the lines of symmetry in your NEM1 textbook (do not copy to your homework paper). Write your answer to part (xii) on your homework paper.
- Do Problem 6 on page 298 of NEM1.

2.4 Parallelograms, Rhombuses and Trapezoids

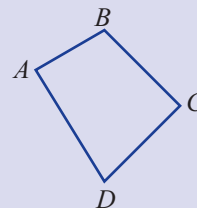
According to the school definition given on page 36, a quadrilateral is a figure enclosed by 4 straight segments. In special quadrilaterals, such as parallelograms, the angle measures and side lengths are related in various ways. In grades 4–6 children learn about these relationships and begin applying them to solve problems. Real geometry begins!

The school definition of quadrilateral uses simple language that children can understand, but the school definition is ambiguous. Students are likely to wonder whether the interior is

part of the quadrilateral, and whether a triangle with a fourth vertex along a straight edge is considered to be triangle or a quadrilateral. As with triangles, these perplexities are resolved by giving the correct mathematical definition.

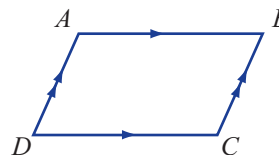
DEFINITION 4.1. A *quadrilateral* consists of 4 distinct points A, B, C, D in the plane, no three of which are collinear, and four segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ that intersect only at their endpoints.

The segments are called the sides and the endpoints are called the vertices of the quadrilateral.

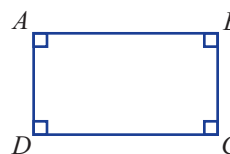


Elementary students are introduced to six types of special quadrilaterals.

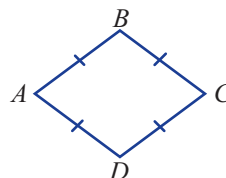
A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.



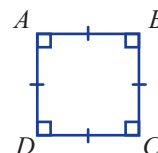
A **rectangle** is a parallelogram all of whose angles are right angles.



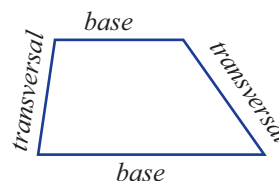
A **rhombus** is a parallelogram with all sides of equal length.



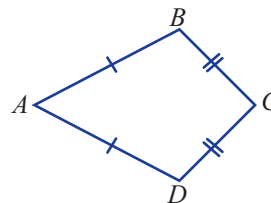
A **square** is a rectangle with all sides of equal length.



A **trapezoid** is a quadrilateral in which one and only one pair of opposite sides are parallel. The parallel sides are called the *bases* of the trapezoid and the other two sides are called *transversals*.



A **kite** is a quadrilateral in which two consecutive sides have equal length and the remaining two sides also have equal length.



High school textbooks use slightly different definitions of rectangle and rhombus. They replace the condition “is a parallelogram” by the more general phrase “is a quadrilateral” and proceed to give proofs that a quadrilateral with four right angles must be a parallelogram, and that a quadrilateral with four sides of equal length also must be a parallelogram. But these proofs are unnecessary complications at the stage when children are first introduced to quadrilaterals. Thus elementary school definitions of rectangle and rhombus include the words “is a parallelogram” for pedagogical, not logical, reasons. In Chapter 4 we will revisit this issue at the grade 7 level.

Several of these terms may apply to one figure. For example, a square is automatically a rectangle and is also a parallelogram. Standardized tests often have questions that require students to know the above definitions and be able to give counterexamples. Here is such a question.

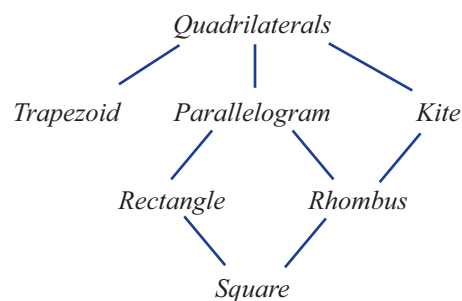
EXERCISE 4.2. a) *Is every parallelogram a kite? Why or why not?*
b) *Is every rhombus a kite? Why or why not?*

Solution: a) No. A ‘thin’ parallelogram is not a kite.



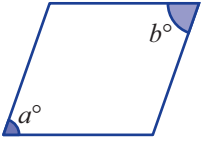
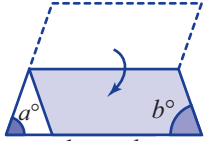
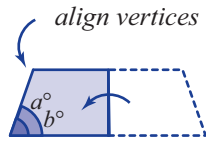
b) Yes. Every rhombus has four equal sides, therefore adjacent sides are equal.

Teaching Comment. The chart on the right shows how the various quadrilaterals are related. Students should not memorize this chart! The point of questions like those in Exercise 4.2 is to get students to learn and appreciate the importance of precise definitions, and to reason with those definitions. Memorization defeats this teaching goal.



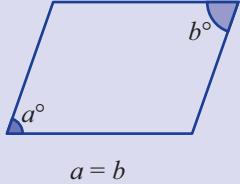
Angles in Parallelograms

If you know the measure of one angle in a parallelogram, then you can determine all of the other angles in the parallelogram. This is commonly presented as two separate facts, both introduced by folding and cutting activities.

 <p>Make a paper parallelogram. Color opposite angles.</p>	 <p>align edges</p> <p>Fold in half.</p>	 <p>align vertices</p> <p>Fold in half again.</p>	<p>What do you notice?</p> <p>$a = b.$</p>
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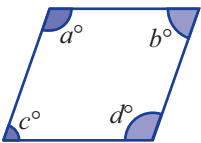
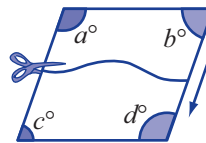
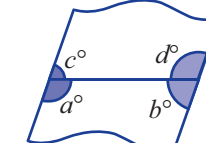
This gives convincing evidence for a simple fact about the angles in a parallelogram:

Opposite angles in a parallelogram are equal.
 (Abbreviation: **opp. ∠s // -ogram.**)



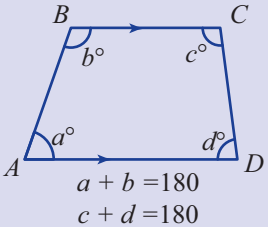
$a = b$

Cutting exercises can also be used. The pictures below show a cutting exercise that leads students to a second fact in parallelograms and trapezoids.

 <p>Make a paper parallelogram.</p>	 <p>Cut and slide down.</p>	 <p>Align edges.</p>	<p>What do you notice?</p> <p>$a + c = 180,$ $b + d = 180.$</p>
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EXERCISE 4.3. Try the cutting exercise above for a trapezoid. Some pairs of angles add up to 180° – which ones?

Interior angles between two parallel sides in a trapezoid (or a parallelogram) are supplementary.
 (Abbreviation: **int. ∠s, $\overline{BC} \parallel \overline{AD}$.)**

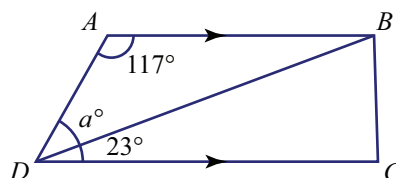


$a + b = 180$
 $c + d = 180$

As a general principle, folding is better than cutting because the figure is not destroyed; after students see that angles match they can easily unfold and interpret the matching in terms of the original figure.

Both facts can be combined with the facts of the previous sections to solve “unknown angle” problems.

EXAMPLE 4.4. The picture shows a trapezoid. Find angle a .



Solution:

$$\begin{aligned}(a + 23) + 117 &= 180 && \text{int. } \angle\text{s, } \overline{AB} \parallel \overline{DC} \\ a + 140 &= 180 \\ a &= 40.\end{aligned}$$

Homework Set 8

- To stay in practice, make the following conversions.
 - 18 m 67 cm = _____ m
 - 1 km 230 m = _____ m
 - 86 mm = _____ cm
 - 4 m 7 cm = _____ cm
 - 23.8 cm = _____ mm
- This exercise provides practice in creating “instructional counter-examples”. Draw:
 - a rectangle that is not a square.
 - a parallelogram that is not a rhombus.
 - a kite that is not a parallelogram.
 - a trapezoid with two opposite sides that clearly have different lengths.
 - a parallelogram whose diagonals are clearly not perpendicular.
- Is this figure a kite? (Read the definition of kite carefully).
- (Study the Textbook!) The basic facts about parallelograms and rhombuses are introduced on pages 68-71 of Primary Math 5B.
 - Write down precise definitions for *parallelogram*, *rhombus* and *trapezoid*.
 - What fact is illustrated on page 69? (see the pictured “student helper”). The illustrations show how to match up angles by cutting, _____, and sliding.
 - What fact is illustrated on page 70?
 - Write answers for Problems 3 and 4.
 - Problem 5 uses the fact established in Problem _____, now applied to a trapezoid instead of a parallelogram.
 - Copy the picture in Problem 5 and show how to cut and slide to see each of the two facts needed to solve the problem. Can you show both in one picture?
 - Write the answers to Problem 6.
- Read pages 72–75 of Primary Math 5B. These give students practice in geometric constructions with ruler, set square, and compass. Using those tools, do the constructions called for in Problems 3 and 5 on page 75.
- Now go to Primary Math 6B. Work through pages 62-65, answering Problems 1-7 in your book, then copying the answers onto your HW paper. Take time to understand the pictures. Notice how the questions take you through the intermediate steps *in order*.
- Continuing in Primary Math 6B, answer Problems 1-5 of Practice 5A on page 66.
- Continuing, answer Problems 1-5 on page 67.
- Draw a kite and label the vertices A, B, C, D .
 - Draw and name a line of symmetry.
 - Describe a folding exercise that shows this symmetry.
 - The symmetry shows which two angles are equal?
 - The line of symmetry bisects which two angles?

10. In general, how many lines of symmetry do each of the following figures have? Draw sketches showing the lines of symmetry.
- isosceles triangle
 - kite
 - rectangle
 - parallelogram
 - square
 - circle.
11. In general, does a parallelogram have rotational symmetry of order 2? of order 4? (see page 42) Draw a sketch.

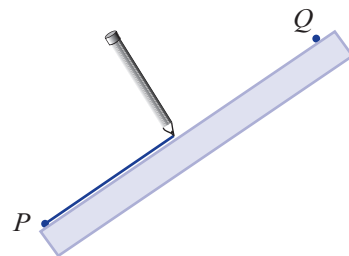
2.5 Geometric Constructions

A part of geometry is concerned with techniques for constructing accurate diagrams. A **geometric construction** is a theoretically exact method of determining points, lines and figures that meet specified conditions. Constructions are traditionally made using two tools: a straightedge and a compass. In school, geometric constructions can be presented as challenging puzzles that provide insight into principles of geometry. They are included in almost every K-12 curriculum around the world.

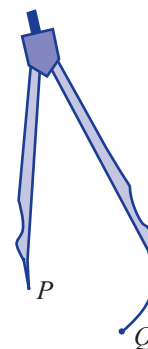
A straightedge is a ruler without any markings and a compass is a tool for drawing circles with any desired center and radius. These tools do not measure lengths or angles, yet they are enough to draw *most* geometric figures. The eight constructions presented in this section underlie much of what will be done in later chapters.

Straightedge and compass constructions involve repeatedly applying two moves according to these instructions: Given two points P and Q , we can

- Use a straightedge to draw the segment \overline{PQ} .
(Abbreviation: **Draw \overline{PQ} .**)



- Use a compass to draw the circle with center P and radius PQ .
(Abbreviation: **Draw circle: center P , radius PQ .**)

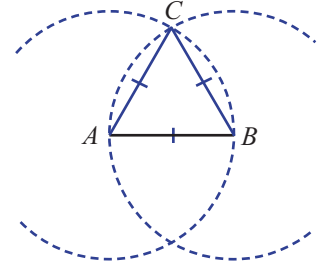


These moves can be used to create new points, which we can label before proceeding (Abbreviation: **Mark point C where...**). In carrying out a compass move, we usually draw only the relevant part of the circle needed for the construction.

For a first example, imagine that someone gives you a piece of paper with a line segment \overline{AB} drawn on it. Your task is to draw an equilateral triangle that has \overline{AB} as one side.

CONSTRUCTION 1. *Construct an equilateral triangle with a given side \overline{AB} .*

- Draw circle: center A , radius AB .
- Draw circle: center B , radius AB .
- These circles intersect at two points. Mark one of them C .
- Draw \overline{AC} and \overline{BC} .

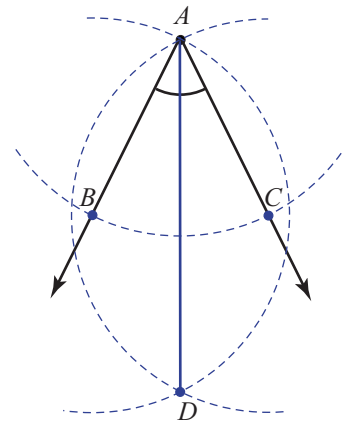


$\triangle ABC$ is an equilateral because all three sides have equal length.

Recall that “bisecting an angle” means drawing a ray that splits the angle into two congruent adjacent angles.

CONSTRUCTION 2. *Bisect a given angle with vertex A .*

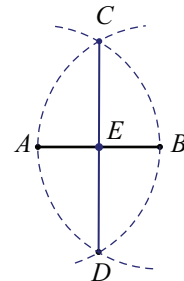
- Draw circle: center A , any radius. Mark points B and C where the circle intersects the sides of the angle.
- Draw circle: center B , radius AB .
- Draw circle: center C , radius AC .
- These circles intersect at A and at another point, D .
- Draw \overline{AD} .



Why is this a bisector? K-8 teachers can give a simple explanation: line \overline{AD} is a line of symmetry, so the two halves of $\angle A$ have equal measure.

CONSTRUCTION 3. *Bisect a given segment \overline{AB} .*

- Draw circle: center at A , radius AB .
- Draw circle: center B , radius AB . Mark the points C and D where the circles intersect.
- Draw \overline{CD} and mark the point E where it intersects \overline{AB} .

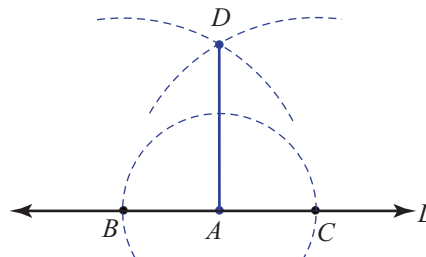


By symmetry, E is the midpoint of \overline{AB} and \overline{CD} is perpendicular to \overline{AB} . Line \overline{CD} is called the *perpendicular bisector* of segment \overline{AB} .

The next construction is the key to building squares and rectangles. The steps are almost the same steps as in the previous construction.

CONSTRUCTION 4. *Given a point A on a line L , construct the line perpendicular to L at A .*

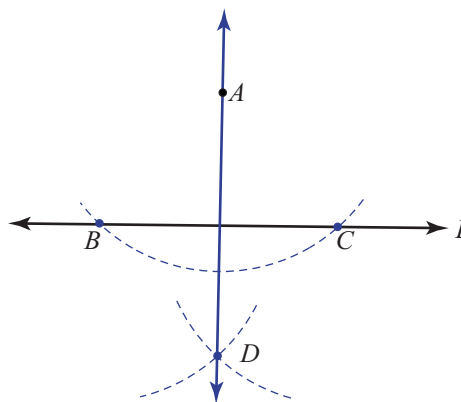
- Draw circle: center A , any radius > 0 . Mark points B and C where the circle intersects line L .
- Using Construction 1, construct an equilateral triangle $\triangle BCD$.
- Draw \overline{AD} .



The next construction is similar, but with the given point A *not* on the given line.

CONSTRUCTION 5. *Given a point A not on a line L , construct the line perpendicular to line L passing through A .*

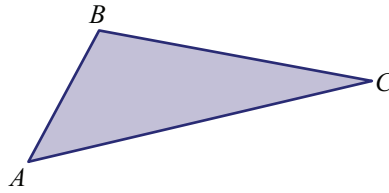
- Mark some point B on L .
- Draw circle: center A , radius AB . This intersects L at B and at another point C .
- Draw circle: center B , radius AB and circle: center C , radius AB . These intersect at A and at another point D .
- Draw \overline{AD} .



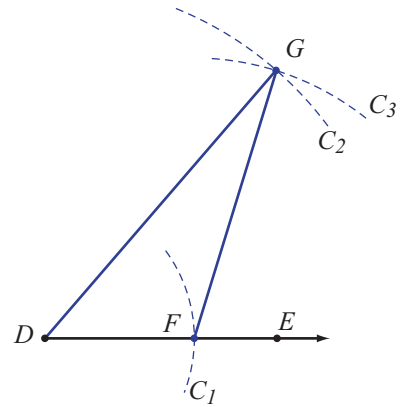
Alternatively, we can draw perpendicular lines using a set square, as we did in Section 2.1. Constructions 4 and 5 show that a set square, while convenient, is not necessary—anything done with the set square can also be done with a compass and straightedge, albeit with more steps.

One can also use a compass and a straightedge to make copies of a given triangle.

CONSTRUCTION 6. Copy a triangle $\triangle ABC$ onto a specified side of a given ray \overrightarrow{DE} .



- Draw circle C_1 : center D , radius AB . Mark F where the circle intersects ray \overrightarrow{DE} .
- Draw circle C_2 : center D , radius AC .
- Draw circle C_3 : center F , radius BC . Mark G at the intersection of circles C_2 and C_3 on the desired side of \overleftrightarrow{DE} .
- Draw segments \overline{DG} and \overline{FG} .



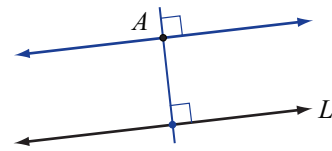
CONSTRUCTION 7. Copy a given angle onto a given ray.

This construction is Problem 7 in the homework for this section. The idea is to make the given angle part of a triangle, then duplicate the triangle using Construction 6.

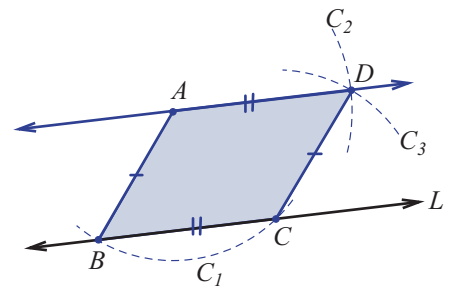
Construction problems can often be solved in several ways. Here is an example.

CONSTRUCTION 8. Construct the line parallel to a given line L through a given point A .

This was done in Section 2.1 using a set square and straightedge to create the picture on the right. Alternatively, one can create the same picture using only a straightedge and a compass by combining Constructions 4 and 5. But it is simpler to proceed as follows.



- Choose a point on L and label it B .
- Draw circle C_1 : center A , radius AB . This intersects L at B and at another point C .
- Draw circle C_2 : center A , radius BC .
- Draw circle C_3 : center C , radius AB . This intersects C_2 at a point D .
- Draw \overleftrightarrow{AD} .



Then $ABCD$ is a parallelogram because its opposite sides have equal length (as will be proved in Chapter 4).

EXERCISE 5.1. Construction 8 can be shortened: only two circles are needed. Write down such a shortened construction.

Homework Set 9

Build your own straightedge by folding a piece of construction paper in half twice. Use this tool (not a ruler!) in Problems 1 and 2. After that, you may use your ruler as a straightedge.

- On your HW paper, mark two points A and B about 2 inches apart. Then do these steps:
 - Draw \overline{AB} . There is nothing more you can do with your straightedge, so pick up your compass.
 - With the information you have — just two points — you can draw only two circles. Draw them and mark all points of intersection.
 - There is nothing more you can do with your compass, so pick up your straightedge. Draw all the lines that your marked points determine.

Which of the constructions described in this section can you see within your final drawing?
- On your HW paper, draw a line L and mark a point P on that line. Using a straightedge and compass, follow the steps of Construction 4 of this section to draw a line through P perpendicular to L .
- On your HW paper, draw a line L and mark a point P **not** on that line. Using a straightedge and compass, follow the steps of Construction 5 of this section to draw a line through P perpendicular to L .
- Using straightedge and protractor, draw a 63° angle.
 - Bisect your angle using straightedge and compass (no protractor!).
- Using ruler and protractor, draw $\triangle PQR$ with $PQ = 6$ cm, $\angle P = 60^\circ$ and $\angle Q = 50^\circ$.
 - Using straightedge and compass (no protractor!), draw a second copy of your triangle.
- Using straightedge and compass:
 - Copy segment \overline{AB} below onto your HW paper.
 - Construct an isosceles triangle with base \overline{AB} whose legs are congruent to segment \overline{CD} .
 - List your steps using the format used to describe constructions in this section.
- In this problem you will supply the details of Construction 7.
 - Draw an angle and label its vertex A . Then, elsewhere, draw a ray \overrightarrow{PQ} .
 - Using straightedge and compass, construct ray \overrightarrow{PR} so that $\angle QPR$ has the same measure as your angle with vertex A . (Experiment on scrap paper until you see what to do.)
 - List your steps using the format used to describe constructions in this section.
- On a blank sheet of paper, mark two points A and C with $AC = 11$ cm. Draw two circles with radius 6 cm: one with center A and one with center C . Complete your figure into a rhombus $ABCD$. Using scissors, cut out your rhombus and use it to answer the following questions:
 - Name two lines of symmetry.
 - Describe a folding exercise that shows these symmetries.
 - Are the diagonals perpendicular? Do they bisect one another? Use folding and symmetry to explain your answer.
- (Study the Textbook!) Read pages 276-277 of NEM 1. Then:
 - Do Problem 1 on page 278 (using ruler, compass and set square).
 - Is the figure a rectangle? Why?
 - What fact about such figures does this suggest?

