6 Replicators

After watching a substantial number of cellular automaton dynamics, eventually a particular time dependent pattern catches the eye. A configuration of occupied sites makes copies of itself, then the copies make copies of themselves, which move toward one another and also toward the boundaries of the evolution. This continues as long as there is room for the evolution. When the innermost copies collide, they annihilate one another. However, the outermost copies continue to reproduce, provided that no occupied sites from the outside impede. The pattern repeats, ad infinitum.

We first saw this kind of evolution, which we call a *replicator*, for the exactly θ LtL rule, where θ is less than or equal to the range. Unexpectedly, we began seeing similar configurations, with slight variations, in many different subregions of LtL space. Then we learned of a very intriguing example, through an electronic newsgroup on cellular automata [CAN]. We refer to this example which exists for High Life, a variant of the Life rule, as *bow tie pasta.* We saw more such examples, which convinced us that this behavior is not exclusive to LtL-like rules, on Christopher Langton's computer at the Santa Fe Institute .

What is required to guarantee the dynamics generated by a replicator? One must begin with a finite configuration of occupied sites which we refer to as Λ. Also required is the time it takes for Λ to make copies of itself; we call this τ.

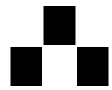
What follows is a detailed description of how one can rescale space in such a way that there is an isomorphism between the space-time diagram of a replicator and a generalization of Pascal's Triangle Mod 2 in an appropriate dimension. If the reader predicts an onset of nausea, she is invited to skip to the examples and pictures in Section 6.3 (this will probably do the convincing anyway).

6.1 Preliminaries.

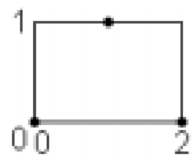
For the following, we assume range ρ box neighborhoods. Thus, in dimension d the neighborhood is a hypercube with side length $2\rho + 1$. The entire discussion generalizes to other types of neighborhoods provided an appropriate tile shape and mapping can be found.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite configuration of $1's$.

- Let $\mathcal{R} \subset \mathbb{R}^d$ be the smallest hyperrectangle containing Λ in such a way that the sites of Λ necessarily lie on the boundary of $\mathcal{R}$. For example, if

$\Lambda =$  $\subset \mathbb{Z}^2,$

then $\mathcal{R}$ is the 2×1 rectangle,



$\subset \mathbb{R}^2$

and, in this case, all of the sites of Λ lie on the boundary of $\mathcal{R}$ (as indicated by the points).

- Orient $\mathcal{R}$ so that it has a vertex at the origin and lies in the region of $\mathbb{R}^d$ where all of the coordinates are greater than or equal to zero. That is,

$$\mathcal{R} = \{x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d : 0 \leq x_i \leq \lambda_i,\ i = 1, 2, ..., d\},$$

where $\lambda_1 \times \lambda_2 \times ... \times \lambda_d$ are the dimensions of $\mathcal{R}$. Since $\Lambda \subset \mathcal{R}$, the *center* of Λ is the centroid

$$(\tfrac{\lambda_1}{2}, \tfrac{\lambda_2}{2}, ..., \tfrac{\lambda_d}{2})$$

of $\mathcal{R}$, and hence, may not lie on Λ. In the example above, the center of Λ is $(1, 0.5)$.

- Let $\mathcal{I}$ be a $\sigma_1 \times \sigma_2 \times ... \times \sigma_d$ *hyper-rectangle* in $\mathbb{R}^d$, large enough to contain Λ and centered at

$$(\tfrac{\lambda_1}{2}, \tfrac{\lambda_2}{2}, ..., \tfrac{\lambda_d}{2}).$$

Tile $\mathbb{R}^d$ with $\mathcal{I}$ so that all of the tiles are identically oriented and the initial tile has center

$$(\tfrac{\lambda_1}{2}, \tfrac{\lambda_2}{2}, ..., \tfrac{\lambda_d}{2}).$$

- Let

$$\mathcal{S} = \sigma_1\mathbb{Z} \times \sigma_2\mathbb{Z} \times ... \times \sigma_d\mathbb{Z} + (\tfrac{\lambda_1}{2}, \tfrac{\lambda_2}{2}, ..., \tfrac{\lambda_d}{2}).$$

Identify the tiles with their centers:

$$(\sigma_1 n_1 + \tfrac{\lambda_1}{2}, \sigma_2 n_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d n_d + \tfrac{\lambda_d}{2}) \in \mathcal{S},$$

for some $(n_1, n_2, ..., n_d) \in \mathbb{Z}^d$, and denote these by $\mathcal{I}_{(n_1,n_2,...,n_d)}$. Then

$$\mathcal{I} \equiv \mathcal{I}_{(0,0,...,0)},$$

but we will use $\mathcal{I}$, or simply write, the tile, when referring to a tile whose location and pattern (see below) are arbitrary.

- Define $\phi : \mathcal{S} \to \mathbb{Z}^d$ by

$$\phi(x_1, x_2, ..., x_d) = (\tfrac{x_1 - \frac{\lambda_1}{2}}{\sigma_1}, \tfrac{x_2 - \frac{\lambda_2}{2}}{\sigma_2}, ..., \tfrac{x_d - \frac{\lambda_d}{2}}{\sigma_d}).$$

Then

$$\phi(\sigma_1 n_1 + \tfrac{\lambda_1}{2}, \sigma_2 n_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d n_d + \tfrac{\lambda_d}{2}) = (n_1, n_2, ..., n_d) \in \mathbb{Z}^d.$$

Thus,

$$\phi(\mathcal{I}_{(n_1,n_2,...,n_d)}) = (n_1, n_2, ..., n_d).$$

- Suppose that each tile is one of two patterns -- painted with Λ, or all $0's$. In the first case, orient the Λ pattern inside the tile so that they have the same center, and denote this by

$$\mathcal{I}^{\Lambda}_{(n_1,n_2,...,n_d)}.$$

(Note that some of the sites of Λ may lie on the boundary of $\mathcal{I}$.) If, on the other hand, $\mathcal{I}_{(n_1,n_2,...,n_d)}$ is painted with all $0's$, denote it by

$$\mathcal{I}^{0}_{(n_1,n_2,...,n_d)}.$$

Use $\mathcal{I}^{\Lambda}$ to denote an arbitrary tile which is painted with Λ, and $\mathcal{I}^0$ one that is painted with all $0's$.

- We have already determined that

$$\phi(\mathcal{I}_{(n_1,n_2,...,n_d)}) = (n_1, n_2, ..., n_d).$$

This tells us the physical location of the image of $\mathcal{I}_{(n_1,n_2,...,n_d)}$. We also need to determine whether this location should contain a 0 or a 1. If the tile is painted with Λ, place a 1 at the point $(n_1, n_2, ..., n_d)$, otherwise place a 0 there. The notation we use to express this is:

$$\phi(\mathcal{I}^l_{(n_1,n_2,...,n_d)}) = \begin{cases} (n_1, n_2, ..., n_d)^1, \text{ if } l = \Lambda \\ (n_1, n_2, ..., n_d)^0, \text{ if } l = 0 \end{cases}.$$

If $\mathcal{I}_{(n_1,n_2,...,n_d)}$ and $\mathcal{I}_{(m_1,m_2,...,m_d)}$, $((n_1, n_2, ..., n_d), (m_1, m_2, ..., m_d) \in \mathbb{Z}^d)$ are separated by k_i tiles in direction i, then

$$|\sigma_i(m_i - (n_i + 1))| = k_i\sigma_i.$$

Thus, their images will be separated by $|m_i - (n_i + 1)| = k_i$ sites in direction i, in $\mathbb{Z}^d$.

Properties of the mapping $\phi : \mathcal{S} \to \mathbb{Z}^d$. We first define, $\cdot$, so that it is a group operation on

$$\mathcal{S} = \sigma_1\mathbb{Z} \times \sigma_2\mathbb{Z} \times ... \times \sigma_d\mathbb{Z} + (\tfrac{\lambda_1}{2}, \tfrac{\lambda_2}{2}, ..., \tfrac{\lambda_d}{2}).$$

Then we show that ϕ is an isomorphism.

- Let x, $y \in \mathcal{S}$. Then

$$x = (\sigma_1 n_1 + \tfrac{\lambda_1}{2}, \sigma_2 n_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d n_d + \tfrac{\lambda_d}{2}),$$

for some $(n_1, n_2, ..., n_d) \in \mathbb{Z}^d$ and

$$y = (\sigma_1 m_1 + \tfrac{\lambda_1}{2}, \sigma_2 m_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d m_d + \tfrac{\lambda_d}{2}),$$

for some $(m_1, m_2, ..., m_d) \in \mathbb{Z}^d$. Define $\cdot$ by

$$x \cdot y \equiv (\sigma_1(n_1 + m_1) + \tfrac{\lambda_1}{2}, \sigma_2(n_2 + m_2) + \tfrac{\lambda_2}{2}, ..., \sigma_d(n_d + m_d) + \tfrac{\lambda_d}{2}) \in \mathcal{S}.$$

- $\mathcal{S}$ is clearly closed under $\cdot$.
- $\cdot$ satisfies the associative law.
- $\mathcal{S}$ has an identity element: $(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, ..., \frac{\lambda_d}{2})$.
- For every element $(\sigma_1 n_1 + \frac{\lambda_1}{2}, \sigma_2 n_2 + \frac{\lambda_2}{2}, ..., \sigma_d n_d + \frac{\lambda_d}{2})$ in $\mathcal{S}$, there exists an inverse

$$(\sigma_1(-n_1) + \tfrac{\lambda_1}{2}, \sigma_2(-n_2) + \tfrac{\lambda_2}{2}, ..., \sigma_d(-n_d) + \tfrac{\lambda_d}{2})$$

  which is also in $\mathcal{S}$.

- $\phi(x)$ is a homomorphism from $\mathcal{S}$ to $\mathbb{Z}^d$ since

  $\phi(x \cdot y) = (n_1 + m_1, n_2 + m_2, ..., n_d + m_d) = (n_1, n_2, ..., n_d) + (m_1, m_2, ..., m_d)$
  $= \phi(x) \cdot \phi(y)$ (the latter $\cdot$ represents the group operation on $\mathbb{Z}^d$).

- ϕ is surjective since, given $b = (n_1, n_2, ..., n_d) \in \mathbb{Z}^d$, there exists

$$a = (\sigma_1 n_1 + \tfrac{\lambda_1}{2}, \sigma_2 n_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d n_d + \tfrac{\lambda_d}{2}) \in \mathcal{S}$$

  such that $\phi(a) = b$.

- ϕ is injective since for x, $y \in \mathcal{S}$ with $x \neq y$, we have

$$x = (\sigma_1 n_1 + \tfrac{\lambda_1}{2}, \sigma_2 n_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d n_d + \tfrac{\lambda_d}{2})$$

  for some $(n_1, n_2, ..., n_d) \in \mathbb{Z}^d$ and

$$y = (\sigma_1 m_1 + \tfrac{\lambda_1}{2}, \sigma_2 m_2 + \tfrac{\lambda_2}{2}, ..., \sigma_d m_d + \tfrac{\lambda_d}{2})$$

for some $(m_1, m_2, ..., m_d) \in \mathbb{Z}^d$ so, for some $k \in \{1, 2, ..., d\}$, $n_k \neq m_k$. But then $\phi(x) = (n_1, n_2, ..., n_d) \neq (m_1, m_2, ..., m_d) = \phi(y)$. $\square$

Definition 6.1.1. The d-dimensional *Pascal-generating CA* rule, ζ_t with neighbor set $\mathcal{N}$, has as its space-time diagram, a generalization (that depends on $\mathcal{N}$) of Pascal's Triangle Mod 2, provided $\zeta_0 = \vec{0}$. Let us define ζ_t, and relate it to Pascal's Triangle, following the $d = 1$ example that Durrett constructs in [Dur], section 5d. Let $\mathcal{N}$ be the neighborhood of $\vec{0} \in \mathbb{Z}^d$. Define

$$\zeta_{t+1}(x) = \sum_{z \in x+\mathcal{N}} \zeta_t(x+z)\text{mod}2.$$

Now assume that $x, y \in \mathbb{Z}^d$. We define

$$f_0(y) = \begin{cases} 1 \text{ if } y = \vec{0} \\ 0 \text{ otherwise} \end{cases} \quad \text{and } f_{t+1}(x) = \sum_{z \in x+\mathcal{N}} f_t(x+z).$$

f generates the $(d+1)$-dimensional generalization of Pascal's Triangle, with neighbor set $\mathcal{N}$, as long as we ignore the $0's$. Also,

$$\zeta_t^{\{0\}}(x) = f_t(x) \text{ mod}2.$$

When we refer to the space-time diagram of the Pascal-generating CA, we mean the above generalization of Pascal's Triangle. Thus, in visualizing it, we assume that the $0's$ have been ignored, prior to modding out by 2. Doing this acts like a shift of the state space at alternate times. Hence, the space-time diagram consists of the space where ζ_t lives at even times, and the same space shifted at odd times. For example, in dimension d with

$$\mathcal{N} = \{(z_1, z_2, ..., z_d) : z_i \in \{-1, 1\}, i = 1, 2, ..., d\},$$

this is equivalent to omitting the values of $\zeta_t^{\{0\}}(z)$ for all $z = (z_1, z_2, ..., z_d)$ such that $t + z_i$ is odd (for some $i \in \{1, 2, ..., d\}$), or $|z| > t$ ($|z| = |z_1| + ... + |z_d|$). Hence, $\zeta_{2t} \subset \mathbb{Z}^d$, $\zeta_{2t+1} \subset \mathbb{Z}^d + (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, $t = 0, 1, 2, ...$ This is illustrated in the examples below.

Example 6.1.1. $d = 1$, $\mathcal{N} = \{\pm 1\}$. We depict a few time steps of the action of f (empty sites have not yet seen any occupied sites in their neighborhoods) and then we show the modded out version which is Pascal's Triangle Mod 2.

time																					
0									...	0	1	0	...								
1								...	0	1	0	1	0	...							
2							...	0	1	0	2	0	1	0	...						
3						...	0	1	0	3	0	3	0	1	0	...					
4					...	0	1	0	4	0	6	0	4	0	1	0	...				
5				...	0	1	0	5	0	10	0	10	0	5	0	1	0	...			
6			..	0	1	0	6	0	15	0	20	0	15	0	6	0	1	0	...		
7		...	0	1	0	7	0	21	0	35	0	35	0	21	0	7	0	1	0	...	
8	...	0	1	0	8	0	28	0	46	0	70	0	46	0	28	0	8	0	1	0	...

Ignoring the $0's$ and modding out by 2 gives:

time																		
0									1									$\subset \mathbb{Z}$
1								1		1								$\subset \mathbb{Z}+1/2$
2							1		0		1							$\subset \mathbb{Z}$
3						1		1		1		1						$\subset \mathbb{Z}+1/2$
4					1		0		0		0		1					$\subset \mathbb{Z}$
5				1		1		0		0		1		1				$\subset \mathbb{Z}+1/2$
6			1		0		1		0		1		0		1			$\subset \mathbb{Z}$
7		1		1		1		1		1		1		1		1		$\subset \mathbb{Z}+1/2$
8	1		0		0		0		0		0		0		0		1	$\subset \mathbb{Z}$

Example 6.1.2. $d = 2$, $\mathcal{N} = \{(z_1, z_2) \in \mathbb{Z}^2 : z_i \in \{1, -1\}, i = 1, 2\}$. The space-time diagram of this CA is what we will usually be referring to when discussing LtL 2-dimensional replicators that have box neighborhoods. We depict a few time steps of the action of f (empty sites have not yet seen any occupied sites in their neighborhoods) and then we show the modular version.

				⋮				
				0				
		...	0	1	0	...		
				0				
				⋮				

			1		1			
				0				
			1		1			

		1		2		1		
			0		0			
		2		4		2		
			0		0			
		1		2		1		

$time = 0$ $\quad$ $time = 1$ $\quad$ $time = 2$

	1		3		3		1	
		0		0		0		
	3		9		9		3	
		0		0		0		
	3		9		9		3	
		0		0		0		
	1		3		3		1	

$time = 3$

1		4		6		4		1
	0		0		0		0	
4		16		24		16		4
	0		0		0		0	
6		24		36		24		6
	0		0		0		0	
4		16		24		16		4
	0		0		0		0	
1		4		6		4		1

$time = 4$

After making the deletions, we mod out by 2 to get:

$$1 \quad \subset \mathbb{Z}^2 \qquad \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \subset \mathbb{Z}^2 + (\tfrac{1}{2}, \tfrac{1}{2}) \qquad \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \subset \mathbb{Z}^2$$

$time = 0$ $\quad$ $time = 1$ $\quad$ $time = 2$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \subset \mathbb{Z}^2 + (\tfrac{1}{2}, \tfrac{1}{2}) \qquad \begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array} \subset \mathbb{Z}^2$$

$time = 3$ $\quad$ $time = 4$

Example 6.1.3. $d = 2$, $\mathcal{N} = \{ \pm e_i,\ i = 1, 2\}$. The space-time diagram of this CA is known as *the* 3-dimensional generalization of Pascal's Triangle Mod 2. Observe that, by rotating (the modded out version of) this example 45° in the counterclockwise direction, we get the previous example. More formally, the mapping, $g(x, y) = (-y + x, y + x)$ is an isomorphism from this example to the previous one. We depict a few time steps of the action of f (empty sites have not yet seen any occupied sites in their neighborhoods) and then we show the modular version.

				⋮				
				0				
		...	0	1	0	...		
				0				
				⋮				

$time = 0$

				1				
			1	0	1			
				1				

$time = 1$

				1				
			2	0	2			
		1	0	4	0	1		
			2	0	2			
				1				

$time = 2$

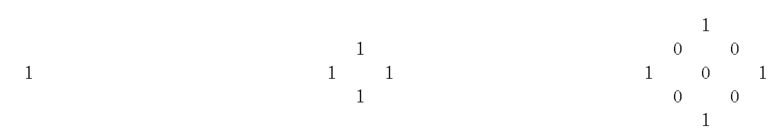
				1				
			3	0	3			
		3	0	9	0	3		
	1	0	9	0	9	0	1	
		3	0	9	0	3		
			3	0	3			
				1				

$time = 3$

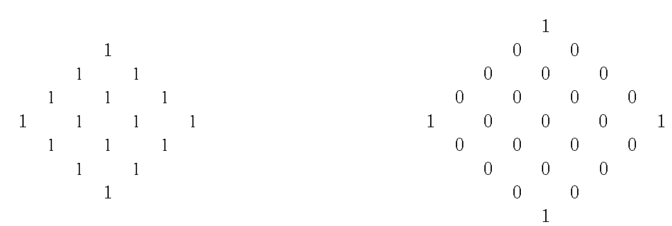
				1				
			4	0	4			
		6	0	16	0	6		
	4	0	24	0	24	0	4	
1	0	16	0	36	0	16	0	1
	4	0	24	0	24	0	4	
		6	0	16	0	6		
			4	0	4			
				1				

$time = 4$

Ignoring the $0's$ and modding out by 2 gives:



$time = 0$ $time = 1$ $time = 2$



$time = 3$ $time = 4$

Let us look at the differences between the space-time diagrams of the d-dimensional Pascal-generating CAs with neighborhoods,

$\mathcal{N} = \{(z_1, z_2, ..., z_d) : z_i \in \{-1, 1\}, i = 1, 2, ..., d\}$, and, $\mathcal{N} = \{\pm e_i,\ i = 1, 2, ..., d\}$, respectively. The previous two examples show that in dimension 2 the space-time diagrams are isomorphic (assuming we ignore the sites which are always 0). The 1-dimensional versions of these neighborhoods are the same and hence the space-time diagrams are the same. However, the 3 or higher dimensional versions of these neighborhoods contain different numbers of neighbors,

$$|\mathcal{N} = \{(z_1, z_2, ..., z_d) : z_i \in \{-1, 1\}, i = 1, 2, ..., d\}| = 2^d,$$

while

$$|\mathcal{N} = \{\pm e_i, i = 1, 2, ..., d\}| = 2d.$$

Thus, the space-time diagrams contain different numbers of occupied sites at each time step. We are mostly interested in the first type of neighborhood, since it is relevant to LtL rules with box neighborhoods. However, our results also apply to the second type of neighborhood provided we make slight modifications. Thus, in what follows, we make the distinction between the neighborhoods and provide the modifications when necessary.