

# Prescribing Ramification

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ABSTRACT. This paper considers groups that can occur as Galois groups of covers of the projective line with prescribed ramification over a subset of the branch locus. The authors have shown that if  $G$  is a finite group with a cyclic-by- $p$  subgroup  $H$ , then over an algebraically closed field of characteristic  $p$ , there exists a Galois cover with Galois group  $G$  of the projective line such that the inertia group over one point is  $H$ . The techniques allow us to also specify the higher ramification groups over this point (Theorem 3.4). Using this result and some Galois theory, we obtain covers defined over a *finite field* with prescribed arithmetic and geometric Galois groups as well as prescribed ramification groups over some points (Theorem 4.5). The results over finite fields relate to a conjecture of Abhyankar and work of Guralnick and Saxl on exceptional covers.

Let  $k$  be an algebraically closed field, and let  $X$  be a projective  $k$ -curve. Prescribing ramification of a cover of  $X$  is an important part of understanding the algebraic fundamental group  $\pi_1(U)$  of open subsets  $U$  of  $X$ . Recall that  $\pi_1(U)$  is defined to be the inverse limit of the inverse system of groups occurring as Galois groups of étale Galois covers of  $U$ . In the case that the field has characteristic zero then the fundamental group is completely understood. Let  $g$  be the genus of  $X$ . Then for  $U = X - \{\tau_1, \dots, \tau_r\}$ , the group  $\pi_1(U)$  is the profinite completion of the topological fundamental group  $\pi_1^{\text{top}}(X - \{\tau_1, \dots, \tau_r\})$ , and this latter group is just the group on  $2g + r$  generators,  $\{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r\}$ , subject to the relation  $\prod_{i=1}^g [a_i, b_i] \prod_{j=1}^r c_j = 1$ . A finite quotient  $\pi_1(U) \rightarrow G$  corresponds to a Galois cover  $Y \rightarrow X$  with Galois group  $G$  and branch locus  $\Delta = \{\tau_1, \dots, \tau_r\}$ . Moreover, the inertia group of a point over  $\tau_i$  is conjugate to the subgroup of  $G$  generated by the image of  $c_i$  in  $G$ . So, in this case we know exactly which groups  $G$  will occur as Galois groups of ramified covers of  $X$ , and we know that all the inertia groups will be cyclic.

For  $k$  of characteristic  $p > 0$ , the situation is quite different. In particular, there can be non-cyclic ramification. For example, consider the cover of the projective line given by the following algebraic extensions: Let  $n|(p-1)$  and take  $k(z) \subset k(w) \subset k(v)$  where  $w^n = z$  and  $v^p - v = w$ . This induces a cover  $V \rightarrow \mathbb{P}_k^1$  with Galois group isomorphic to a cyclic-by- $p$  group:  $P \rtimes C$  where  $P$  is a  $p$ -group and

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$C = \mathbb{Z}/n\mathbb{Z}$ . The inertia group at 0 is  $C$  and at  $\infty$  is  $P \rtimes C$ . In fact, the inertia group of any ramified point of a Galois cover of a  $k$ -curve  $X$  is cyclic-by- $p$ . Notice that over the complex numbers this cover cannot exist.

The main result in this area is Harbater and Raynaud's proof of the Abhyankar Conjecture [Ha94], [Ra94] (see also [Ha99] for improved version of Harbater's result). This determines exactly which finite groups occur as Galois groups of  $X$  with branch locus  $\Delta \neq \emptyset$ . Moreover, the result shows that any such group could exist as a Galois group of a cover of  $X$  in such a way that the ramification over all but one of the points of  $\Delta$  is tame (hence cyclic). The result does not prescribe the ramification, however for a cover of the line, ramified only at infinity, it does show that the  $p$ -part of the inertia can be taken to be a  $p$ -Sylow subgroup. Subsequent results of Harbater and Pop have shown the existence of covers with small  $\Delta$  (see [Ha95] and [P]). In Section 3 of this paper, we take a group  $G$  with cyclic-by- $p$  subgroups  $H_i$ , and we build a cover with Galois group  $G$  in such a way that the subgroups  $H_i$  occur amongst the inertia groups of the cover (see Theorem 3.4, similar results have been obtained by Harbater [Ha99]).

In Section 4 of this paper we consider covers over finite fields. Let  $k$  be the algebraic closure of  $\mathbb{F}_p$ . Over  $\mathbb{F}_p$ , Abhyankar has conjectured that a group  $G$  will occur over  $\mathbb{A}_{\mathbb{F}_p}^1$  if and only if it occurs over  $\mathbb{P}_k^1 - \{2 \text{ points}\}$ . For any group  $A$  let  $p(A)$  be the subgroup of  $A$  generated by its  $p$ -Sylow subgroups. This is a normal subgroup. Suppose that a group  $A$  is generated by  $p(A)$  together with one other element  $x$ . Then by the result of Harbater and Raynaud, this group will occur over  $\mathbb{P}_k^1 - \{2 \text{ points}\}$ . In Theorem 4.5, we show that for *some* finite field extension  $l$  of  $\mathbb{F}_p$ , there exists a Galois cover of  $\mathbb{A}_l^1$  with arithmetic Galois group  $A$  and geometric Galois group  $p(A)$ .

One of the motivations for writing this paper was to answer a question from a paper of Guralnick and Saxl [GS] regarding covers with prescribed Galois groups and inertia groups. In their paper the authors consider an analogy between group theoretic properties and properties of covers. For example, assume that  $k$  is an algebraically closed field of positive characteristic and  $f : Y \rightarrow X$  is a branched cover of degree  $n$  (not necessarily Galois) with a totally ramified point. Let  $G$  be the Galois group of the Galois closure of the cover  $f$  and let  $Z$  denote the corresponding curve. We call  $G$  the *monodromy group* of  $f : Y \rightarrow X$ . Let  $N$  be the subgroup of  $G$  corresponding to  $Y$ , and  $H$  the inertia group of a point in  $Z$  over the totally ramified point of  $X$ . Then  $G$  is a transitive group of degree  $n$  (acting on the cosets of  $N$ ). Since  $H$  is an inertia group, we know that  $H$  is a cyclic-by- $p$  group. Completing at the totally ramified point, shows that  $H$  is transitive in the permutation representation on the cosets of  $N$ . Thus,  $G = HN$ .

Moreover, in the above situation  $G$  is a *primitive* permutation group if and only if the cover  $f : Y \rightarrow X$  is indecomposable (i.e. it does not factor through another non-trivial cover). This is equivalent to saying that  $N$  is a maximal subgroup of  $G$ . In [GS], the authors classify all primitive transitive permutation groups with transitive cyclic-by- $p$  subgroups.

Using our result Proposition 4.1, we can show that groups of the type considered in [GS] occur as monodromy groups of covers with a totally ramified point. More specifically, given any transitive permutation group  $G$  with point stabilizer  $N$  and  $H$  a subgroup of  $G$  that is cyclic modulo a normal  $p$ -subgroup, there exists a cover  $f : X \rightarrow \mathbb{P}^1$  with monodromy group  $G$  and inertia group  $H$  at  $\infty$ . We actually

prove a stronger result where we need not assume  $k$  is algebraically closed. (See Theorem 5.1). In particular, this shows that the list of possible monodromy groups of indecomposable covers with a totally ramified point given in [GS] is precise.

The structure of the paper is as follows. The first three sections work over an algebraically closed field of positive characteristic, and the last two work over arbitrary perfect fields. Sections 1 and 2 prepare the ground work for the results Section 3. The main result is Theorem 3.4 in Section 3, and it concerns covers with prescribed ramification. Section 4 proves a result related to a conjecture of Abhyankar over finite fields, and Section 5 relates work of Guralnick and Saxl and others on exceptional covers.

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**Notation** For a finite group  $G$ , an element (resp. subgroup) of order prime to  $p$  will be called a  $p'$  element (resp.  $p'$  subgroup). Following [HS], a cover  $Y \rightarrow X$  is a morphism of schemes that is finite and generically separable. If  $G$  is a finite group, then a  $G$ -Galois cover is a cover  $Y \rightarrow X$  (possibly disconnected) together with a homomorphism  $\iota : G \rightarrow \text{Aut}_X(Y)$ , with respect to which  $G$  acts simply transitively on each generic geometric fiber of the cover. A  $G$ -Galois cover which is also connected will be called a *Galois cover with group  $G$*  (or just Galois if we do not care about the group).

## 1. $H$ Covers

Throughout this section let  $H$  be a semi-direct product of a  $p$ -group  $P$  and a cyclic prime to  $p$  group ( $p'$  group)  $C$ . Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . We build a smooth Galois cover of  $\mathbb{P}_k^1$  with Galois group isomorphic to  $G$  such that the inertia group of one ramified point is isomorphic to  $H$ . We use a result of Katz in order to build a cover of the projective  $k$ -line with Galois group  $H$  which is ramified in two points, one tamely and the other totally. First we need to mention some background on inertia groups and higher ramification groups.

Pick a compatible set of  $n$ th roots of unity  $\zeta_n \in k$  for all positive integers  $n$  prime to  $p$ . That is, take a set of  $n$ th roots of unity so that  $\zeta_{nm}^m = \zeta_n$  for all positive integers  $n$  and  $m$  prime to  $p$ . Suppose that we have a Galois cover  $Y \rightarrow X$  of curves and a branch point  $\tau \in X$ . Let  $s$  be the ramification index of a point  $\hat{\tau}$  over  $\tau$ . If  $(s, p) = 1$  then we call the ramification *tame* and otherwise we call it *wild*. If the ramification is tame then the inertia group of  $\hat{\tau}$  is cyclic and we have a canonical generator: Pick a uniformizer  $x$  at  $\tau$  on  $X$  and take an element  $y$  in the complete local ring  $\hat{\mathcal{O}}_{Y, \hat{\tau}}$  such that  $y^s = x$ . Then the *canonical generator of the inertia* of  $\hat{\tau}$  over  $\tau$  is the element  $c$  of the inertia group which takes  $y$  to  $\zeta_s y$  (this is independent of the choice of  $x$  and  $y$ ). Given a ramified cover  $Y \rightarrow X$  with *tame* branch locus  $\tau_1, \tau_2, \dots, \tau_r$  such that the canonical generator of inertia over  $\tau_i$  is  $c_i$ , the  $n$ -tuple  $(c_1, c_2, \dots, c_r)$  is called the *description* of the cover. It is determined up to conjugation of the individual  $c_i$ 's (this corresponds to the choice of  $\hat{\tau}_i$  over  $\tau_i$ ). If the point  $\hat{\tau}$  is a wildly ramified point, then its inertia group will be  $P \rtimes C$  where  $P$  is a  $p$  group and  $C$  is a cyclic  $p'$  group of order  $n$ . We can choose  $x$  and  $y$  as above,

and let  $c \in C$  be the element such that  $c(y) = \zeta_n y$ . Call  $c$  the *canonical generator of the tame part of the inertia group over  $\tau$* .

In addition to the inertia group, we also have higher ramification groups. Again let  $Y \rightarrow X$  be a Galois cover with branch point  $\tau \in X$  and  $\hat{\tau}$  a point lying over  $\tau$ . Let  $\Gamma$  be its Galois group. For  $i > 0$  let

$$D_i = \{g \in \Gamma \mid g(\alpha) = \alpha \pmod{y^{i+1}} \forall \alpha \in \hat{\mathcal{O}}_{Y, \hat{\tau}}\}.$$

Then  $D_0$  is just the inertia group of  $\hat{\tau}$  over  $\tau$  and  $D_i = \{1\}$  for some sufficiently large  $i$ . The  $D_i$  form a descending chain of subgroups. Given an inertia group  $D_0$  it is not true that each possible descending chains of subgroups arises as a chain of higher ramification groups of some ramified point. In fact, it is unknown exactly which chains are possible, however if we only look at the *distinct* higher ramification groups (namely, ignoring the  $i$  for which  $D_i = D_{i+1}$ ), then Maus has shown in [Ms] exactly which groups occur.

We will call a set of groups  $N_0, N_1, \dots, N_m$  an *allowable set of higher ramification groups* if (by Maus) there exist an extension  $L$  over  $k((t))$  such that the distinct higher ramification groups of a point over the closed point in  $\text{Spec}(k[[t]])$  are the  $N_i$ .

In [Kz88], Katz finds an equivalence of categories between étale covers of  $\text{Spec}(k((\frac{1}{x})))$  and what he calls “special” covers of the projective line. More specifically, let  $x$  be the local parameter at 0. Then the complete local ring of  $\mathbb{P}_k^1$  at infinity is isomorphic to  $k((\frac{1}{x}))$ , and we have the natural morphism  $\iota : \text{Spec}(k((\frac{1}{x}))) \rightarrow \mathbb{P}_k^1$ . Given any cover of the projective line  $Y \rightarrow \mathbb{P}_k^1$ , we can take the pullback of it over the morphism  $\iota$ , to obtain a (possibly disconnected) cover  $\hat{Y} \rightarrow \text{Spec}(k((\frac{1}{x})))$ . This gives us a functor from the category of covers of  $\mathbb{P}_k^1$  to the category of covers of  $\text{Spec}(k((\frac{1}{x})))$ . It is called the inverse image functor.

We call a Galois cover of the projective line *special* if it is étale away from the set  $\{0, \infty\}$ , tamely ramified over 0, totally ramified over infinity, and if its Galois group has a unique  $p$ -Sylow subgroup. The Katz-Gabber result, [Kz88, Main Theorem], shows that the inverse image functor from the category of *special* covers of the projective line to the category of étale covers of  $\text{Spec}(k((\frac{1}{x})))$  is an equivalence of categories.

**PROPOSITION 1.1.** *Let  $H = P \rtimes \langle h \rangle$  where  $h$  is a  $p'$  element of  $H$ , and let  $H = N_0, N_1, \dots, N_m$  be a descending chain of allowable higher ramification groups. Then there exists a Galois cover  $Y_H \rightarrow \mathbb{P}_k^1$  with Galois group  $H$ , ramified only at 0 and  $\infty$ , such that it is totally ramified over  $\infty$ , and tamely ramified over 0 with canonical generator of inertia  $h$ . Moreover, the distinct higher ramification groups over  $\infty$  are isomorphic to the  $N_i$ .*

**PROOF.** The result of Katz and Gabber, [Kz88, Main Theorem], implies that for any étale cover  $\hat{W} \rightarrow \text{Spec}(k((\frac{1}{x})))$  there exists a special cover  $Y \rightarrow \mathbb{P}_k^1$  such that if  $\hat{Y} \rightarrow \text{Spec}(k((\frac{1}{x})))$  is the pullback over  $\iota$  then there is an isomorphism  $\phi : \hat{Y} \rightarrow \hat{W}$  of covers of  $\text{Spec}(k((\frac{1}{x})))$ . Moreover, if  $\hat{W}$  is connected (hence Galois) then the cover  $Y \rightarrow \mathbb{P}^1$  will be Galois, totally ramified over infinity, and the inertia group of the point lying over  $\infty$  in  $\mathbb{P}_k^1$  will be isomorphic to the Galois group of the cover  $\text{Spec}(L) \rightarrow \text{Spec}(k((\frac{1}{x})))$ . Thus the higher ramification groups will also be isomorphic, as will the canonical generator of the tame part of the inertia over infinity.

By hypothesis, there exists an extension  $k(\left(\frac{1}{x}\right)) \subset L$  such that the Galois group (inertia group) is  $H$ , the distinct higher ramification groups are  $N_i$   $i = 0, 1, \dots, m$  and the canonical generator of inertia on the tame part of the inertia group is  $h^{-1}$ . Thus, the corresponding cover  $\text{Spec}(L) \rightarrow \text{Spec}(k(\left(\frac{1}{x}\right)))$  is an étale Galois cover. This cover will have a corresponding special Galois cover  $Y \rightarrow \mathbb{P}_k^1$  with Galois group  $H$ , totally ramified over infinity. The canonical generator of inertia at 0 must be the inverse of the canonical generator of the tame part of the inertia at infinity (since when we mod out by  $P$  we get a tamely ramified cover of the projective line branched at two points). Thus it must be  $h$ . This gives the proposition.  $\square$

## 2. $G - H$ Covers

In this section we use the previous result to build a Galois cover with Galois group  $G$  and inertia group  $H$  under an assumption on the ramification. We use formal patching to ‘paste’ together the Galois cover with Galois group  $H$ , from the previous section, to a  $G$ -Galois cover. This original  $G$ -Galois cover is assumed to have a tamely ramified point whose canonical generator of inertia equals the inverse of a generator of the  $p'$ -cyclic part of  $H$ . After patching, the tame ramification on the original cover is replaced by a wildly ramified point with inertia group  $H$ .

Again, take  $k$  to be an algebraically closed field of characteristic  $p > 0$ . Suppose we have a group  $G$  with subgroup  $N$ . Suppose also that there exists an  $N$ -Galois cover  $Y \rightarrow X$  with branch locus  $D \subset X$ . We define  $\text{Ind}_N^G Y \rightarrow X$  to be the (possibly disconnected) cover of  $X$  consisting of  $[G : N]$  distinct copies of  $Y$  indexed by the left cosets of  $N$  in  $G$ . We identify  $Y$  with the identity copy of  $Y$  in the induced cover. The cover is  $G$ -Galois by letting  $G$  act on the cover as it does on the cosets. The branch locus of the new cover is still  $D$ .

**THEOREM 2.1.** *Let  $G$  be any group with subgroup  $H$  such that  $H = P \rtimes C$  where  $P$  is a  $p$ -group and  $C$  is a cyclic  $p'$  group generated by an element  $h$ . Let  $G_0$  be a subgroup such that  $G = \langle H, G_0 \rangle$ . Suppose that there exists a Galois cover  $Y_0 \rightarrow X_0$  of smooth connected projective  $k$ -curves with Galois group  $G_0$  and branch locus  $\Delta_{Y_0}$ . Assume that there is a tamely ramified point  $\eta_0 \in \Delta_{Y_0}$  with canonical generator of inertia  $h^{-1}$ . Let  $\phi : W_0 = \text{Ind}_{G_0}^G Y_0 \rightarrow X_0$ . Then there exists a smooth connected projective  $k$ -curve  $X^*$  of genus  $g = g(X_0)$  and a Galois cover  $\psi : Y^* \rightarrow X^*$  with Galois group  $G$  and branch locus  $\Delta_{Y^*}$  such that:*

- (1) *For some  $\delta \in \Delta_{Y^*}$  there exists  $\sigma \in \psi^{-1}(\delta)$  with inertia group  $H$ .*
- (2) *Let  $\Delta'_{Y^*} = \Delta_{Y^*} - \{\delta\}$  and  $\Delta'_{Y_0} = \Delta_{Y_0} - \{\eta_0\}$ . There are bijections  $\iota : \Delta'_{Y^*} \rightarrow \Delta'_{Y_0}$  and  $\hat{\iota} : \psi^{-1}(\Delta'_{Y^*}) \rightarrow \phi^{-1}(\Delta'_{Y_0})$  such that  $\phi \circ \iota = \hat{\iota} \circ \psi$  and that the higher ramification groups of a point  $\eta \in \phi^{-1}(\Delta'_{Y_0})$  are equal to the higher ramification groups of the point  $\hat{\iota}(\eta) \in \psi^{-1}(\Delta'_{Y^*})$ .*

*Moreover, if  $N_m \subset N_{m-1} \subset \dots \subset N_1 = H$  is an allowable set of higher ramification groups then the higher ramification groups of  $\sigma$  may be assumed to be isomorphic to the  $N_j$ 's.*

**REMARK 2.2.** In the result above, a more complicated argument shows that  $X^*$  can be taken to be equal to  $X_0$ . However, since in most of the applications we will take  $X_0$  to be a projective line, it does not seem to be worth the substantial lengthening of the proof.

Before we prove this result we recall a patching result from [HS]. Consider a group  $\Gamma$  and a Galois cover  $Y \rightarrow X$  of semi-stable curves with Galois group isomorphic to  $\Gamma$ . Let  $\Delta$  be the branch locus of this cover,  $S$  be the singular locus of  $X$ , and  $\tilde{Y} \rightarrow \tilde{X}$  be the pullback of  $Y \rightarrow X$  over the normalization  $\tilde{X} \rightarrow X$ . Suppose that  $\tilde{Y}$  is smooth. The cover  $Y \rightarrow X$  is called *admissible* if for each point  $\eta$  lying over a point in  $S$ , the cover is tame at  $\eta$  and the canonical generators of inertia at the two points  $\eta_1, \eta_2 \in \tilde{Y}$  over  $\eta$  are inverses in  $G$ . Notice that the admissibility condition is automatically satisfied if  $S \cap \Delta = \emptyset$ . Let  $D \subset X$  be a finite set of smooth points. Let  $\pi_A^{\text{adm}}(X - D)$  denote the set of (isomorphism classes of) finite groups  $G$  such that there exists a Galois admissible cover of  $X$  with Galois group  $G$  which is étale over  $X - (S \cup D)$ . In the case that  $X$  is smooth, then this is just the set  $\pi_A(X - D)$  of (isomorphism classes of) finite Galois groups of étale Galois covers of  $X - D$ .

**THEOREM 2.3.** [HS, Cor. 1] *Let  $X$  be a connected projective semi-stable  $k$ -curve of arithmetic genus  $g$ , with singular locus  $S$ . Let  $\Delta' \subset X - S$  be a (possibly empty) finite set of closed points of  $X$ .*

- (1) *Then there is a smooth projective  $k$ -curve  $X^*$  of genus  $g$  and a finite set  $\Delta'^*$  of  $k$ -points of  $X^*$  with  $|\Delta'| = |\Delta'^*|$ , such that  $\pi_A^{\text{adm}}(X - \Delta') \subset \pi_A(X^* - \Delta'^*)$ .*
- (2) *Given a connected admissible Galois cover  $\phi : Y \rightarrow X$  étale off  $S \cup \Delta'$  with Galois group  $G$ , there exists a smooth connected projective  $k$ -curve  $X^*$  of genus  $g$  and a smooth connected Galois cover  $\phi^* : Y^* \rightarrow X^*$  with Galois group  $G$  and branch locus  $\Delta'^*$ , such that  $|\Delta'| = |\Delta'^*|$ . Moreover, there exist bijections  $\rho : \Delta' \rightarrow \Delta'^*$  and  $\hat{\rho} : \phi^{-1}(\Delta') \rightarrow \phi^{*-1}(\Delta'^*)$  such that  $\phi^* \circ \hat{\rho} = \rho \circ \phi$  and that the higher ramification groups of a point  $\eta \in \phi^{-1}(\Delta')$  are isomorphic to the higher ramification groups of the point  $\hat{\rho}(\eta) \in \phi^{*-1}(\Delta'^*)$ .*

**PROOF.** Notice that (1) follows from (2). In [HS, Cor. 1] the Galois cover  $\phi^* : Y^* \rightarrow X^*$  with Galois group  $G$  of part (2) is built by deforming the cover  $\phi : Y \rightarrow X$ . The cover  $\phi^*$  is obtained from the cover  $\phi$  by first locally deforming  $\phi$  over  $k[[t]]$ . Away from  $S$ , this deformation is trivial. Then formal patching is used to obtain a global deformation. Given a point  $\eta \in \phi^{-1}(\Delta')$ , let  $\delta = \phi(\eta)$ . The deformation of the cover  $\phi$  near  $\eta$  is constructed to be the trivial deformation, so the higher ramification groups remain constant in the deformation. By the Bertini-Noether theorem [FJ, Proposition 9.29], we obtain the desired cover over  $k$ . Thus in the resulting cover  $\phi^*$  there is a unique point  $\eta^*$  which corresponds to the deformation and then specialization of  $\eta$ , and its higher ramification groups will be isomorphic to those of  $\eta$ . This correspondence defines  $\rho$ . By [HS, Cor. 2] the genus of  $X^*$  is  $g$ .  $\square$

Now we can proceed with the proof of the main result.

**PROOF.** (Proof of 2.1) Let  $X_1$  be a projective  $k$ -line. By Proposition 1.1 there exists a Galois cover  $Y_1 \rightarrow X_1$  with Galois group  $H$  ramified only at  $\Delta_{Y_1} = \{\eta_1, \delta_1\}$ , such that it is tamely ramified over  $\eta_1$  with canonical generator of inertia  $h$ , and totally ramified over  $\delta_1$  with higher ramification groups isomorphic to the  $N_j$ 's.

Consider  $W_1 = \text{Ind}_H^G Y_1 \rightarrow X_1$ . The branch locus of this cover is still  $\Delta_{Y_1} = \{\eta_1, \delta_1\}$ . We want to patch this induced  $G$ -Galois cover to the  $G$ -Galois cover  $\phi : W_0 \rightarrow X_0$  in such a way that the resulting cover is connected.

Let  $X$  be the union of  $X_0$  and  $X_1$  with  $\eta_1$  identified with  $\eta_0$  at a point  $\eta$  in such a way that  $X_0$  has a normal crossing at  $\eta$  and is smooth away from  $\eta$ . By hypothesis, there exists  $\xi_0 \in \phi^{-1}(\eta_0)$  with canonical generator of inertia  $h^{-1}$ . By construction, there is a point  $\xi_1 \in Y_1$  with canonical generator of inertia  $h$ . Let  $Y$  be the union of  $W_0 = \text{Ind}_{G_0}^G Y_0$  and  $W_1 = \text{Ind}_H^G Y_1$  such that for each  $g \in G$   $g(\xi_0)$  is identified with  $g(\xi_1)$  at a normal crossing (see [HS] for details). Let  $\xi$  be the point on  $Y$  obtained by identifying  $\xi_0$  with  $\xi_1$ . This gives a  $G$ -Galois cover  $\psi : Y \rightarrow X$ , of connected singular curves.

The only singularity on  $X$  is the normal crossing  $\eta$  and the only singularities on  $Y$  are the  $g(\xi)$ 's. Thus  $\psi : Y \rightarrow X$  is a  $G$ -Galois cover of semi-stable curves. Let  $\tilde{X}$  be the normalization of  $X$ . This is the disjoint union of  $X_0$  and  $X_1$ . Let  $\tilde{\psi} : \tilde{Y} \rightarrow \tilde{X}$  be the pullback of  $\psi : Y \rightarrow X$  over  $\tilde{X} \rightarrow X$ . Then  $\tilde{Y}$  is disjoint union of  $[G_0 : G]$  copies of  $Y_0$  and  $[H : G]$  copies of  $Y_1$ . The cover  $\tilde{\psi}|_{Y_0}$  is the cover  $Y_0 \rightarrow X_0$  and  $\tilde{\psi}|_{Y_1}$  is the cover  $Y_1 \rightarrow X_1$  above. Thus for every  $g \in G$  the two points on  $\tilde{Y}$  lying over  $g(\xi) \in Y$  are  $g(\xi_1)$  and  $g(\xi_0)$  with canonical generators of inertia  $ghg^{-1}$  and  $gh^{-1}g^{-1} = (ghg^{-1})^{-1}$ , respectively. This cover is connected as  $G = \langle H, G_0 \rangle$ . Thus the  $G$ -cover  $\psi : Y \rightarrow X$  is admissible. Let  $\Delta_Y$  denote the branch locus of  $\psi : Y \rightarrow X$ . Then  $\Delta_Y = \Delta'_{Y_0} \cup (\Delta_{Y_1} - \{\eta_0\}) \cup \{\eta\}$  and the inertia groups remain the same. More precisely, we have bijections  $\alpha : \Delta_{Y_0} - \{\eta_0\} \rightarrow \Delta_Y - \{\delta_1, \eta\}$  and  $\hat{\alpha} : \phi^{-1}(\Delta_{Y_0} - \{\eta_0\}) \rightarrow \psi^{-1}(\Delta_Y - \{\delta_1, \eta\})$  such that  $\psi \circ \hat{\alpha} = \alpha \circ \phi$  and the higher ramification groups (including inertia groups) of a point  $\gamma \in \phi^{-1}(\Delta_{Y_0} - \{\eta_0\})$  are isomorphic to the higher ramification groups of the point  $\hat{\alpha}(\gamma) \in \psi^{-1}(\Delta_Y - \{\delta_1, \eta\})$ .

By Theorem 2.3 there exists a projective curve of genus  $g = g(X_0)$  and a connected  $G$ -Galois cover  $\psi^* : Y^* \rightarrow X^*$  with branch locus  $\Delta_{Y^*}$  such that  $|\Delta_{Y^*}| = |\Delta'_Y|$  where  $\Delta'_Y = \Delta_Y - \{\eta\}$ . Moreover there exist bijections  $\rho : \Delta'_Y \rightarrow \Delta_{Y^*}$  and  $\hat{\rho} : \psi^{-1}(\Delta'_Y) \rightarrow \psi^{*-1}(\Delta_{Y^*})$  such that  $\psi^* \circ \hat{\rho} = \rho \circ \psi$  and that the higher ramification groups of a point  $\gamma \in \psi^{-1}(\Delta'_Y)$  are isomorphic to the higher ramification groups of the point  $\hat{\rho}(\gamma) \in \psi^{*-1}(\Delta_{Y^*})$ . There exists a point  $\delta_1 \in \Delta'_Y$  such that some  $\sigma_1$  over it has inertia group  $H$  and the higher ramification groups are the  $N_j$ . Thus,  $\sigma_1^* = \hat{\rho}(\sigma_1) \in \Delta_{Y^*}$  has inertia group  $H$  and the higher ramifications are  $N_j$ .

Let  $\rho'$  be the restriction of  $\rho$  to  $\Delta'_Y - \{\delta_1\}$  and  $\hat{\rho}'$  be the restriction of  $\hat{\rho}$  to  $\psi^{-1}(\Delta_{Y^*} - \{\delta_1^*\})$ . Then  $\iota = \rho' \circ \alpha$  and  $\hat{\iota} = \hat{\rho}' \circ \hat{\alpha}$  are the desired bijections for (2).  $\square$

REMARK 2.4. If we eliminate the condition that  $G = \langle H, G_0 \rangle$ , then the resulting covers  $Y \rightarrow X$  and  $Y^* \rightarrow X^*$  would be disconnected.

### 3. Prescribing ramification

We prove a more specific result, Theorem 3.4, regarding covers with prescribed Galois group and inertia groups. Above we were able to prescribe the Galois group and inertia group of a cover under the assumption of the existence of another cover with particular tame ramification at a point. We built the cover by increasing the ramification at that point (roughly speaking). Here we guarantee that this condition is satisfied by adding some ramification, if necessary.

We begin with Proposition 3.1 that adds tame ramification with prescribed canonical generator of inertia to an existing cover. Then we prove a technical result, Proposition 3.2. We finish with the main result, Theorem 3.4. The field  $k$  is assumed to be algebraically closed and of positive characteristic  $p > 0$ .

**PROPOSITION 3.1.** *Let  $X_0$  be a smooth connected projective  $k$ -curve of genus  $g$ . Given a Galois cover  $\phi : Y_0 \rightarrow X_0$  with Galois group  $G$  and branch locus  $\Delta_{Y_0}$ , let  $h$  be any  $p'$  element of  $G$ . Then there exists a smooth connected projective  $k$ -curve  $X^*$  of genus  $g$  and a Galois cover  $\phi^* : Y^* \rightarrow X^*$  with Galois group  $G$ , branch locus  $\Delta_{Y^*}$  and  $\alpha, \tau \in \Delta_{Y^*}$  such that*

- (1) *The ramification over  $\alpha$  is tame, and the canonical generator of inertia of some point  $\hat{\alpha}$  over  $\alpha$  is  $h$ .*
- (2) *The ramification over  $\tau$  is tame, and the canonical generator of inertia of some point  $\hat{\tau}$  over  $\tau$  is  $h^{-1}$ .*
- (3) *There exist bijections  $\rho : \Delta_{Y_0} \rightarrow \Delta'_{Y^*} = \Delta_{Y^*} - \{\alpha, \tau\}$  and  $\hat{\rho} : \phi^{-1}(\Delta_{Y_0}) \rightarrow \phi^{*-1}(\Delta'_{Y^*})$  such that  $\phi^* \circ \hat{\rho} = \rho \circ \phi$  and that the higher ramification groups (including inertia groups) of a point  $\eta \in \phi^{-1}(\Delta_{Y_0})$  are isomorphic to the higher ramification groups of the point  $\hat{\rho}(\eta) \in \phi^{*-1}(\Delta'_{Y^*})$ .*

**PROOF.** Let  $X_1$  be a projective line and let  $Y_1 \rightarrow X_1$  be a  $C = \langle h \rangle$  cover totally ramified over two points  $\alpha$  and  $\tau$  with canonical generators of inertia  $h$  and  $h^{-1}$  respectively. Let  $\Delta_{Y_1} = \{\alpha, \tau\}$ . Let  $\text{Ind}_C^G Y_1 \rightarrow X_1$  be the  $G$ -Galois cover of  $X_1$ . The branch locus of this cover is again  $\Delta_{Y_1} = \{\alpha, \tau\}$ .

Pick  $\eta_0 \in X_0 - \Delta_{X_0}$  and  $\eta_1 \in X_1 - \Delta_{Y_1}$  (unramified points). Let  $X$  be the union of  $X_0$  and the  $X_1$  with  $\eta_0$  identified with  $\eta_1$  at a point  $\eta$  in such a way that  $X$  has a normal crossing at  $\eta$  and is smooth away from  $\eta$ . Identify  $X_i$  with its copy in  $X$ . Pick  $\delta_0 \in Y_0$  lying over  $\eta_0 \in X_0$  and  $\delta_1 \in Y_1$  lying over  $\eta_1 \in X_1$ . Let  $Y$  be the union of  $Y_0$  and the  $\text{Ind}_C^G Y_1$  formed by identifying the points  $g(\delta_0) \in Y_0$  and  $g(\delta_1) \in \text{Ind}_C^G Y_1$  for each  $g \in G$ . In this way we obtain a  $G$ -Galois cover  $Y \rightarrow X$  of semi-stable curves (see [HS] for details), where  $X$  has arithmetic genus  $g$ . Moreover, the cover is admissible since it is unramified on the singular locus. It is connected because the Galois cover  $Y_0 \rightarrow X_0$  is, and the branch locus is  $\Delta = \Delta_{Y_0} \cup \Delta_{Y_1}$ . The inertia groups remain the same. In particular, the canonical generator of some point  $\hat{\tau}$  over  $\tau$  is  $h^{-1}$  and of some point  $\hat{\alpha}$  over  $\alpha$  is  $h$ .

By 2.3 there exists a smooth connected projective  $k$ -curve  $X^*$  of genus  $g$  and a connected  $G$ -Galois cover  $Y^* \rightarrow X^*$  with branch locus  $\Delta_{Y^*}$  such that  $|\Delta_{Y^*}| = |\Delta|$  and such that there exist bijections (note  $\Delta' = \Delta$  here)  $\rho : \Delta \rightarrow \Delta_{Y^*}$  and  $\hat{\rho} : \phi^{-1}(\Delta) \rightarrow \phi^{*-1}(\Delta_{Y^*})$  such that  $\phi^* \circ \hat{\rho} = \rho \circ \phi$  and that the ramification groups of a point  $\xi \in \phi^{-1}(\Delta)$  are isomorphic to the higher ramification groups of the point  $\hat{\rho}(\xi) \in \phi^{*-1}(\Delta_{Y^*})$ . We have that  $\Delta = \Delta_{Y_0} \cup \Delta_{Y_1}$  and the following:

- (1) The inertia group of some point  $\rho(\hat{\alpha}) \in Y^*$  over  $\rho(\alpha) \in X^*$  has the same canonical generator of inertia as  $\hat{\alpha} \in Y_1 \subset Y$  over  $\alpha \in X_1 \subset X$ , and this in turn has canonical generator  $h$ .
- (2) The inertia group of some point  $\hat{\rho}(\hat{\tau}) \in Y^*$  over  $\rho(\tau) \in X^*$  has the same canonical generator of inertia as  $\hat{\tau} \in Y_1 \subset Y$  over  $\tau \in X_1 \subset X$ , and this in turn has canonical generator of inertia  $h^{-1}$ .
- (3) Replacing  $\rho$  by  $\rho|_{\phi^{-1}(\Delta_{Y_0})}$ , we obtain the proposition. □

For any finite group  $G$  recall from the introduction that  $p(G)$  is the subgroup of  $G$  generated by all the  $p$ -Sylow subgroups of  $G$ . It is normal in  $G$ . If  $S$  is a  $p$ -Sylow subgroup of  $G$ , then there exists a  $p'$  subgroup  $F$  of  $N_G(S)$  such that  $F/(F \cap p(G)) = G/p(G)$ . Call such an  $F$  a  $p'$  supplement to  $p(G)$  in  $G$ . Moreover, given any  $p'$  element  $h \in N_G(S)$ , then  $h$  is conjugate to some element of  $F$ . This is a consequence of a result of Schur and Zassenhaus (or see [Ha94] lemma 5.3). Next follows an explicit technical result giving a cover with prescribed Galois group and some prescribed ramification data (however, there is ramification which is not prescribed and over which we have *no* control). Theorem 3.4 is a simplified version of this result.

**PROPOSITION 3.2.** *Let  $G$  be any group with subgroups  $H_1, \dots, H_t$  such that  $H_i$  is the semi-direct product of a  $p$ -group  $P_i$  and a cyclic  $p'$  group  $C_i = \langle h_i \rangle \subset G$ . Let  $S$  be a  $p$ -Sylow subgroup of  $G$ . Assume that for some  $s \leq t$  there exist  $\sigma_1, \dots, \sigma_s \in G$  with  $\{g_1 = \sigma_1^{-1} h_1^{-1} \sigma_1^{-1}, \dots, g_s = \sigma_s h_s^{-1} \sigma_s^{-1}\} \subset F$  where  $F$  is some  $p'$  supplement to  $p(G)$ . Let  $\{g_1, \dots, g_s, g_{s+1}, \dots, g_r\}$  be a generating set of  $F$  such that  $g_1 \dots g_s g_{s+1} \dots g_r = 1$ . Then there exists a smooth projective Galois cover  $Y^* \rightarrow \mathbb{P}_k^1$  with Galois group  $G$  and branch locus  $\Delta_{Y^*} = \{\tau_1, \dots, \tau_t, \alpha_1, \dots, \alpha_{t-s}, \beta_1, \dots, \beta_{r-s-1}, \gamma\}$  where:*

- (1) *For each  $i = 1, \dots, t$ , the inertia group of some point  $\hat{\tau}_i$  over  $\tau_i$  is isomorphic to  $H_i$ ;*
- (2) *For each  $j = 1, \dots, t - s$  the inertia group of some point  $\hat{\alpha}_j$  over  $\alpha_j$  has canonical generator  $h_{s+j}$ ;*
- (3) *For each  $l = 1, \dots, r - s - 1$ , the inertia group of some point  $\hat{\beta}_l$  over  $\beta_l$  has canonical generator  $g_{s+l}$ ;*
- (4) *The inertia group of some point  $\hat{\gamma}$  over  $\gamma$  is contained in  $S \rtimes \langle h' \rangle$  for some element  $h' \in F$ .*

Moreover, if  $N_{i,m_i} \subset N_{i,m_i-1} \subset \dots \subset N_{i,1} = H_i$  is an allowable set of higher ramification groups then the higher ramification groups of  $\hat{\beta}_1$  may be assumed to be isomorphic to the  $N_{i,j}$ 's.

**PROOF.** Since  $F$  is a  $p'$  group and  $g_1 \dots g_s g_{s+1} \dots g_r = 1$ , by [Gr, XIII, Cor. 2.12] there exists a Galois cover  $Y_0 \rightarrow X_0$  with Galois group  $F$  and branch locus  $\Delta_{Y_0}$ , branch description:  $(g_1, \dots, g_r)$ . That is,  $\Delta_{Y_0} = \{\beta_1, \dots, \beta_r\}$  such that for each  $i = 1, \dots, r$ , the inertia group of some point  $\hat{\beta}_i$  over  $\beta_i$  has canonical generator  $g_i$ .

By [Ha99, Prop. 4.1] there exists a smooth connected  $G$ -Galois cover  $Y_1 \rightarrow X_1$  with branch locus  $\Delta_{Y_1} = \{\beta'_1, \dots, \beta'_{r-1}, \gamma\}$  and such that:

- i:** For each  $i = 1, \dots, r - 1$ , the inertia group of some point  $\hat{\beta}'_i$  over  $\beta'_i$  has canonical generator  $g_i$ .
- ii:** The inertia group of some point  $\hat{\gamma}$  over  $\gamma$  is equal to  $S \rtimes \langle h' \rangle$  for some element  $h'$  of order prime to  $p$ .

So this is a  $G$ -Galois cover with description of the tame ramification given by

$$(g_1, \dots, g_s, g_{s+1}, \dots, g_{r-1}).$$

By Proposition 3.1 there exists a Galois cover  $Y_2 \rightarrow X_2$  with Galois group  $G$  and description of the tame ramification given by

$$(g_1, \dots, g_s, h_{s+1}^{-1}, \dots, h_t^{-1}, h_{s+1}, \dots, h_t, g_{s+1}, \dots, g_{r-1}).$$

Since the description is only unique up to conjugation we can also say that  $Y_2 \rightarrow X_2$  is a  $G$ -Galois cover with description of the tame ramification given by

$$(h_1^{-1}, \dots, h_t^{-1}, h_{s+1}, \dots, h_t, g_{s+1}, \dots, g_{r-1}).$$

Then by Theorem 2.1 applied to  $h_1, \dots, h_t$  we are done.  $\square$

REMARK 3.3. Notice that if we take  $t = 1$  in the theorem, there are two cases:  $s = 1$  and  $s = 0$  :

- (1) If  $s = 1$  then  $H = P \rtimes \langle h \rangle$  where  $h \in N_G(S)$ . In this case we are replacing the cover from [Ha99] by another  $G$  cover of  $\mathbb{P}_k^1$  where we increase the inertia groups from  $\langle h \rangle$  to  $H$ .
- (2) If  $s = 0$  then  $H = P \rtimes \langle h \rangle$  where  $h \notin N_G(S)$ . In this case we are adding two branch points to the cover from [Ha99], one tame with canonical generator of inertia  $h$  and the other wild with inertia group  $H$ .

The following result is a weaker version of the proposition above, but in many applications it is sufficient.

THEOREM 3.4. *Let  $G$  be any group with subgroups  $H_1, \dots, H_t$  such that  $H_i$  is the semi-direct product of a  $p$ -group  $P_i$  and a cyclic  $p'$  group  $C_i = \langle h_i \rangle$ . Then there exists a Galois cover  $Y^* \rightarrow \mathbb{P}_k^1$  with Galois group  $G$  and branch locus  $\Delta_{Y^*}$  containing a set  $\{\tau_1, \dots, \tau_t\}$  such that for each  $i = 1, \dots, t$ , the inertia group of some point  $\hat{\tau}_i$  over  $\tau_i$  is equal to  $H_i$ .*

- (1) *If  $N_{i,m_i} \subset N_{i,m_i-1} \subset \dots \subset N_{i,1} = H_i$  is an allowable set of higher ramification groups then the higher ramification groups of  $\hat{\tau}_i$  may be assumed to be equal to the  $N_{i,j}$ 's.*
- (2) *Suppose in addition that  $G$  has a normal subgroup  $N$  such that each  $H_i \subset N$  and such that  $G/N$  is a  $p'$ -cyclic group. Then we may assume that the subcover  $Y_N \rightarrow \mathbb{P}_k^1$  of  $Y^* \rightarrow \mathbb{P}_k^1$  corresponding to  $N \subset G$  is  $p'$ -cyclic, branched at only two points, and totally ramified over each branch point.*

PROOF. The proof of the main part of the theorem and item 1 follows immediately from the previous proposition. For item 2, we use a different argument to build  $Y^* \rightarrow \mathbb{P}_k^1$ . Let  $X_1 = \mathbb{P}_k^1$ .

By [Ha95], there exists a smooth connected Galois cover  $Y_1 \rightarrow X_1$  with group  $G$  with branch locus  $\Delta_{Y_1}$  and such that the sub cover  $Y_{1,N} \rightarrow X_1$  of  $Y_1^* \rightarrow X_1$  corresponding to  $N \subset G$  is  $p'$ -cyclic, branched at only two points,  $\beta_1, \beta_2$ , and totally ramified over each branch point. Let  $\{\beta_1, \dots, \beta_r\} = \Delta_1$  be the branch locus of the cover  $Y_1 \rightarrow X_1$ , and let  $(g_1, \dots, g_r)$  be a tame description of the cover. Then for any  $\hat{\beta} \in Y_1$  lying over  $\beta_i \in \Delta_1$  for  $i > 2$ , the inertia group of  $\hat{\beta}$  lies in  $N$ . By Proposition 3.1 there exists a Galois cover  $Y_2 \rightarrow X_2$  with Galois group  $G$  with description of the tame ramification given by

$$(g_1, \dots, g_r, h_1^{-1}, \dots, h_t^{-1}, h_{s+1}, \dots, h_t).$$

Moreover  $X_2$  is a projective line. Let  $\{\beta'_1, \dots, \beta'_r, \alpha_1, \dots, \alpha_t, \tau_1, \dots, \tau_t\} = \Delta_2$  be the branch locus. By 3.1, the inertia groups over the  $\beta_i$ 's are isomorphic to those over the  $\beta_i$ 's in the cover  $Y_1 \rightarrow X_1$ . The ramification over the  $\alpha_i$ 's and  $\tau_i$ 's is all tame. Then by 2.1 applied to  $h_1, \dots, h_t$  we have a Galois cover  $Y_3 \rightarrow X_3$  with Galois group  $G$  and branch locus  $\Delta_3 = \{\beta''_1, \dots, \beta''_r, \alpha'_1, \dots, \alpha'_t, \tau'_1, \dots, \tau'_t\}$ . The inertia groups are as follows:

- i:** For any  $\hat{\beta}_i'' \in Y_3$  lying over  $\beta_i'' \in \Delta_3$ , the inertia group  $I_i$  is equal to the inertia group of a point  $\hat{\beta} \in Y_1$  over  $\beta_i$ , and if  $i > 2$  then  $I_i$  is contained in the normal subgroup  $N \subset G$ .
- ii:** For any  $\hat{\alpha}_i' \in Y_3$  lying over  $\alpha_i' \in \Delta_3$ , the inertia group  $I_i$  is conjugate to the cyclic subgroup generated by  $h_i \subset H_i \subset N$ . Since  $N$  is normal in  $G$ ,  $I_i \subset N$ .
- iii:** For any  $\hat{\tau}_i' \in Y_3$  lying over  $\tau_i' \in \Delta_3$ , the inertia group  $I_i$  is conjugate to  $H_i \subset N$ . Again, since  $N$  is normal in  $G$ ,  $I_i \subset N$ .

Thus the subcover  $Y_{3,N} \rightarrow X_3$  corresponding to  $N \subset G$  is branched at only  $\beta_1'$  and  $\beta_2'$  and over each it is totally branched. Since  $G/N$  is  $p'$ -cyclic, we see that  $Y_{3,N} \cong \mathbb{P}_k^1$ .  $\square$

#### 4. Arithmetic covers

The results in the previous sections can be used to get information about Galois covers of curves defined over non-algebraically closed fields. Let  $k$  be a perfect field. Let  $X$  be a smooth projective  $k$ -curve. We call a cover of  $k$ -curves  $Y \rightarrow X$  *regular* if the algebraic closure of  $k$  in the function field  $K_Y$  of  $Y$  is  $k$  itself (some authors use ‘geometric’). For algebraically closed fields this is of course always the case, however in general the extension may have some ‘arithmetic’ component.

Let  $k$  be a field with  $k'/k$  a finite Galois extension. Let  $Y$  be a curve defined over  $k'$  and let  $A$  be a subgroup of  $\text{Aut}_k(Y)$  such that  $A$  induces the full group of  $k$ -automorphisms of  $k'$ . Then  $A$  induces a Galois cover  $Y \rightarrow X := Y_A$  with Galois group  $A$  where  $X$  is a curve defined over  $k$ . Let  $G$  be the normal subgroup of  $A$  which is trivial on  $k'$ . So  $A/G \cong \text{Gal}(k'/k)$ . Then there is a Galois cover  $\phi : Y \rightarrow X_{k'} := X \times_k k'$  that is regular (over  $k'$ ) and has Galois group  $G$ . Let  $\Delta$  be the set of branch points of the cover  $Y \rightarrow X$  and note that this set is  $A$ -invariant and so defined over  $k$ . Let  $\Delta_G$  be the set of branch points of  $Y \rightarrow X_{k'}$ . This set is defined over  $k'$ .

Given any Galois cover  $\phi : Y \rightarrow X$  with Galois group  $A$  and branch locus  $\Delta$  defined over  $k$ , let  $k'$  be the algebraic closure of  $k$  in  $K_Y$ . Then  $\phi : Y \rightarrow X$  factors through  $X_{k'} \rightarrow X$ , and  $Y \rightarrow X_{k'}$  is a Galois cover defined over  $k'$ , and we have a short exact sequence

$$1 \rightarrow \text{Gal}(Y/X_{k'}) \rightarrow A \rightarrow \text{Gal}(k'/k) \rightarrow 1.$$

Notice that the algebraic closure of  $k'$  in  $K_Y$  is  $k'$  again. Thus  $Y \rightarrow X_{k'}$  is regular and we call  $G = \text{Gal}(Y/X_{k'})$  the *geometric Galois group* of  $Y \rightarrow X$ . Let  $\Delta_{\bar{k}} = \Delta \times_k \bar{k} = \{\tau_1, \dots, \tau_r\}$ ,  $Y_{\bar{k}} = Y \times_k \bar{k}$  and  $X_{\bar{k}} = X \times_k \bar{k}$ . We say that  $Y \rightarrow X$  has *inertial description*  $(H_1, \dots, H_r)$  if  $Y_{\bar{k}} \rightarrow X_{\bar{k}}$  has the property that over every  $\tau_i$  there exists a  $\hat{\tau}_i$  with inertia group  $H_i$ . This is defined only up to individual conjugation. Notice that if  $\tau \in \Delta$  is a closed point in  $X$  and  $\tau_i$  and  $\tau_j$  in  $X_{\bar{k}}$  are conjugate geometric points over  $\tau$  then the inertia groups of  $\hat{\tau}_i$  and  $\hat{\tau}_j$  over  $\tau$  are  $H_i$  and  $H_j$  respectively and they are conjugate to one another in  $A$ , but not necessarily in  $G$ .

If we begin with a regular non-Galois cover  $Y \rightarrow X$  then its Galois closure  $\bar{Y} \rightarrow X$  may or may not be regular. Let  $A$  and  $G$ , respectively be the arithmetic and geometric Galois groups of the cover  $\bar{Y} \rightarrow X$ , then  $A$  is called the *arithmetic monodromy group* of  $Y \rightarrow X$  and  $G$  the *geometric monodromy group* of  $\bar{Y} \rightarrow X$ .

On the level of fundamental groups, we have an analogous short exact sequence. As before, we define the algebraic fundamental group to be

$$\pi_1(U) := \varprojlim_{Y/X \text{ étale over } U} \text{Gal}(Y/X).$$

Let  $U_{\bar{k}} = U \times_k \bar{k}$ . Then

$$1 \rightarrow \pi_1(U_{\bar{k}}) \rightarrow \pi_1(U) \rightarrow \text{Gal}(\bar{k}/k) \rightarrow 1$$

and it splits if there exists a  $k$ -point on  $U$ . The group  $\pi_1(U_{\bar{k}})$  is called the geometric fundamental group. Much less is known about  $\pi_1(U)$  or even  $\pi_A(U)$  than is known about  $\pi_1(U_{\bar{k}})$  or  $\pi_A(U_{\bar{k}})$ . However, as noted in the introduction Abhyankar has conjectured in [A] that if  $k = \mathbb{F}_p$  then  $\pi_1(U) \twoheadrightarrow A$  if and only if  $\pi_1(U_{\bar{k}} - \{\tau_{r+1}\}) \twoheadrightarrow A$ . In this section, we give a result, Theorem 4.5, related to this conjecture.

**PROPOSITION 4.1.** *Let  $A$  be a finite group with normal subgroup  $G$ , such that there exists a Galois extension  $k'/k$  with  $\text{Gal}(k'/k) = A/G$ . Let  $\phi : Y \rightarrow X$  be a regular Galois cover with Galois group  $A$  and branch locus  $\Delta$  defined over  $k$ . Let  $\Delta_{\bar{k}} = \Delta \times_k \bar{k} = \{\tau_1, \dots, \tau_t\}$  and let  $(I_1, \dots, I_t)$  be an inertial description of  $\phi$ . Let  $Y \rightarrow Y_G$  be the cover corresponding to the subgroup  $G$  of  $A$  and let  $\Delta_G$  be its branch locus. Let  $Z = Y \times_k k'$ . Then there exists a Galois cover  $Z \rightarrow X$  dominating  $\phi$  with Galois group  $A \times A/G$ . Moreover, there is an inclusion  $A \xhookrightarrow{\iota} A \times A/G$ , such that the corresponding intermediate cover  $Z \xrightarrow{\psi} W$  has the following properties:*

- (1) *The cover  $\psi : Z \rightarrow W$  is Galois and has arithmetic Galois group  $A$  and geometric Galois group  $G$  (identifying  $A$  and  $G$  with their copies in  $A \times A/G$  via  $\iota$ );*
- (2) *The intermediate cover  $Z \rightarrow Z_G \rightarrow W$  corresponding to  $G \subset A$  has  $Z_G = Y_G \times_k k'$ , and the branch locus of  $Z \rightarrow Z_G$  is contained in  $\Delta_G \times_k k'$ .*
- (3) *For any branch point  $\beta \in W$ , its inertia group (with respect to  $\psi$ ) is  $A$ -conjugate to  $I_j \cap G$  for some  $j = 1, \dots, t$ .*
- (4) *The cover  $W \rightarrow X$  is regular.*

**PROOF.** Let  $M_1$  and  $K_1$  be the function fields of  $Y$  and  $X$ , respectively. By assumption,  $M_1/K_1$  is a finite Galois extension with Galois group  $A$ , such that  $k$  is algebraically closed in  $M_1$ . Moreover, we know that there exists a finite Galois extension  $k'/k$  with  $A/G = \text{Gal}(k'/k)$ . Let  $B$  denote this quotient group.

We now consider the field  $M = M_1 k'$ . Since  $k'K_1$  and  $M_1$  are linearly disjoint, this extension is Galois with  $\text{Gal}(M/K_1) = A \times B$ , and it corresponds to a cover  $Z \rightarrow X$  with Galois group  $A \times B$  with inertial description  $(I_1, \dots, I_t)$ .

Let  $\pi : A \rightarrow A/G = B$  be the natural projection. Consider the injective group homomorphism  $\iota : A \rightarrow A \times B$  by  $a \mapsto (a, \pi(a))$ . Let  $A_0$  denote the image of  $\iota$ . Thus  $A \cong A_0$ . Let  $K$  be the fixed field  $M^{A_0}$  of  $A_0$  acting on  $M$ .

Since  $k(x)$  is finitely generated over  $k$  and has transcendence degree 1, the same is true for  $K$ . Thus  $K$  is the function field of some curve  $W$  over  $k$  and  $M/K$  corresponds to a cover  $\psi : Z \rightarrow W$ . Note that  $k'$  is the algebraic closure of  $k$  in  $M$ . Let  $G_0$  denote the image of  $G$  in  $A_0$  under  $\iota$ . Since  $\pi(G)$  is trivial,  $G_0$  is contained in the subgroup of  $A_0$  fixing  $Kk'$ . Now  $[Kk' : K] = [k' : k] = [A : G] = [A_0 : G_0]$ , so  $G_0$  is the subgroup of  $A_0$  fixing  $k'$ . Thus  $G_0 \cong G$  is the geometric Galois group of the cover  $\psi : Z \rightarrow W$ . This proves part 1.

For part 2, notice that  $K_1 k' \subset M^{G_0}$ . Now, since  $G_0$  is the image of  $G$ , we have  $M_1^G \subset M^{G_0}$ . Thus  $M_1^G k' \subset M^{G_0}$ . Recall that  $K_1 k' \cap M_1^G = K_1$  so

$\deg(M_1^G k'/M_1^G) = \deg(K_1 k'/K_1) = |A/G|$ . Also,  $\deg(M/M_1^G) = |A/G||G|$ , so now  $\deg(M/M_1^G k') = |G| = |G_0|$ , so  $M_1^G k' = M^G$ . Thus  $Z_G = Y_G \times_k k'$  and the branch locus of the intermediate cover  $Z \rightarrow Z_G$  of  $\psi : Z \rightarrow W$  is contained in  $\Delta_G \times_k k'$ .

For part 3, we consider the inertial description of  $\psi : Z \rightarrow W$ . The cover  $\psi : Z \rightarrow W$  is an intermediate cover of  $Z \rightarrow X$ , so extending constants we have  $Z_{\bar{k}} \xrightarrow{\psi} W_{\bar{k}} \rightarrow X_{\bar{k}}$ . Let  $\beta_{\bar{k}} \in W_{\bar{k}}$  be a branch point of  $\psi : Z_{\bar{k}} \rightarrow W_{\bar{k}}$  such that for some (hence all)  $\hat{\beta}_{\bar{k}} \in \psi^{-1}(\beta_{\bar{k}})$  the inertia group  $I$  of  $\hat{\beta}_{\bar{k}}$  is non-trivial. Let  $\tau_{\bar{k}}$  be the image of  $\hat{\beta}_{\bar{k}}$  under  $Z_{\bar{k}} \rightarrow X_{\bar{k}}$ . Then  $\tau_{\bar{k}}$  must be a branch point of  $Z_{\bar{k}} \rightarrow X_{\bar{k}}$ . Thus  $\tau_{\bar{k}} \in \Delta_{\bar{k}}$ , and the inertia group of  $\hat{\beta}_{\bar{k}}$  in  $Z_{\bar{k}} \rightarrow X_{\bar{k}}$  is  $A$ -conjugate to some  $I_j$ . The group  $G$  is the geometric Galois group of  $\psi : Z \rightarrow W$ , so it is also equal to the Galois group of  $\psi : Z_{\bar{k}} \rightarrow W_{\bar{k}}$ . Thus,  $I = I_j \cap G$ .

For part 4, consider the algebraic closure of  $k$  in  $K$ . Since  $(a, \pi(a))$  acts on  $k'$  as  $\pi(a)$ ,  $A_0$  acts on  $k'$  as  $B$  does. Thus the fixed field of  $A_0$  on  $k'$  is  $k$ , and  $k' \cap K = k$ . This shows that  $W \rightarrow X$  is regular.  $\square$

REMARK 4.2. In the above result, applying the full strength of Theorem 3.4, it is possible to prescribe not only the inertia groups but also the higher ramification groups as well. That is, if we take  $I_{j,m_j} \subset I_{j,m_{j-1}} \subset \dots \subset I_{j,1} = I_j$  to be the higher ramification groups of  $\phi : Y \rightarrow X$  in Proposition 4.1, then in the resulting cover  $Z \rightarrow W$ , the higher ramification groups in 4.1.3 over  $\beta \in W$  will be  $A$ -conjugate to  $I_{j,s} \cap G$ . The results in the rest of this section and those of Section 5 rely on 4.1. In each of these results it is possible to prescribe the higher ramification groups. We have eliminated this step for the sake of brevity.

Let  $F$  any perfect field, and let  $G(F)$  be the set of finite groups which are Galois groups over *any* finite extension  $L$  of  $F$ .

PROPOSITION 4.3. *Let  $F$  be any perfect field and let  $A$  be a finite group with normal subgroup  $G$  such that  $A/G \in G(F)$ . Suppose that there exist subgroups  $I_1, \dots, I_t$  of  $G$  such that each  $I_j$  is a cyclic by  $p$  group. Then there exists a Galois cover  $Z \rightarrow W$  defined over some  $k/F$  such that: The arithmetic Galois group is  $A$ ; the geometric Galois group is  $G$ ; and there exist points  $\hat{\gamma}_i \in Z$  with inertia group  $I_i$ . In particular:*

- (1) *If  $A/G$  is cyclic, then we may take  $F$  a finite field.*
- (2) *If  $A/G$  is cyclic  $p'$ , then we may take  $F$  finite and  $W = \mathbb{P}_k^1$ .*

PROOF. Let  $X_0 = \mathbb{P}_{\bar{F}}^1$  where  $\bar{F}$  is the algebraic closure of  $F$ . By the Abhyankar Conjecture (new proof in [Ha99]), we know that  $A$  is the Galois group of a Galois cover  $\phi_0 : Y_0 \rightarrow X_0$  with branch locus  $\Delta_{Y_0}$ . By Proposition 3.2 we may assume that there exist points  $\hat{\alpha}_i$  lying over points of  $\Delta_{Y_0}$  with inertia group  $I_i$ .

Because this cover corresponds to a Galois extension  $M_0/K_0$  of finite type over  $\bar{F}$ , the extension descends to a finite extension  $k/F$ . That is, there exist  $k/F$  and a Galois field extension  $M_1/k(x)$  with Galois group  $A$  such that  $k$  is algebraically closed in  $M_1$ ,  $K_0 = k(x) \otimes_k \bar{\mathbb{F}}_p$  and  $M_0 = M_1 \otimes_k \bar{\mathbb{F}}_p$ . Let  $K_1 = k(x)$  and let  $\phi_1 : Y_1 \rightarrow X_1 = \mathbb{P}_k^1$  be the cover corresponding to  $M_1/K_1$ . This cover has Galois group  $A$ , and we can choose  $k$  so that the branch locus  $\Delta_{Y_1} \subset X_1$  contains  $k$ -valued points  $\{\delta_1, \dots, \delta_t\}$  with inertia description  $(I_1, \dots, I_t)$ . Moreover, by enlarging  $k$  if necessary, we may assume that  $A/G = \text{Gal}(k'/k)$  for some finite Galois extension  $k'/k$ .

Consider Proposition 4.1 with  $X = X_1$  and  $Y = Y_1$ . Then there exists a Galois cover  $Z \rightarrow X_1$  dominating  $Y_1 \rightarrow X_1$  with Galois group  $A \times A/G$ . Moreover there is an intermediate cover  $Z \xrightarrow{\psi} W \rightarrow X_1$ . By 4.1.1 the cover  $\psi$  has arithmetic Galois group  $A$  and geometric Galois group  $G$ . Part 2 of 4.1 implies that in the intermediate cover  $Z \rightarrow Z_G$  of  $\psi : Z \rightarrow W$ , we have  $Z_G = Y_{1,G} \times_k k'$  and the branch locus of  $Z \rightarrow Z_G$  is contained in  $\Delta_G \times_k k'$ . By 4.1.3, for any branch point  $\beta \in W$ , its inertia group (with respect to  $\psi$ ) is  $A$ -conjugate to  $I \cap G$  where  $I$  is an inertia group of  $Y_1 \rightarrow \mathbb{P}_k^1$ . If  $\beta$  lies over  $\delta_i$ , then its inertia group is  $A$ -conjugate to  $I_i \cap G = I_i$ . This concludes the general part of the corollary.

For part 1, suppose that  $A/G$  is cyclic. Then, clearly  $A/G \in S(F)$  for any finite field. So we can take  $F$  to be a finite field.

For part 2, suppose that  $A/G$  is cyclic  $p'$ -group. By 3.4.2 (applied with  $G$  of 3.4.2 equal to  $A$  here and  $N$  in 3.4.2 equal to  $G$  here) we may assume that the  $A$ -Galois cover  $\phi_0 : Y_0 \rightarrow X_0$  is such that the subcover  $Y_{0,G} \rightarrow X_0$  corresponding to  $G \subset A$ , is branched at only two points  $\gamma_1, \gamma_2 \in \Delta_{Y_0}$  and is totally ramified over each of these points. That implies that for every  $\gamma \in \Delta_{Y_0} - \{\gamma_1, \gamma_2\}$  the inertia group of any point over  $\gamma$  lies in  $G$ . For  $i = 1, 2$ , let  $H_i$  be the inertia group of a point  $\hat{\gamma}_i \in Y_0$  over  $\gamma_i$ , and let  $\gamma'_i \in Y_{0,G}$  be the image of  $\hat{\gamma}_i$  under  $Y_0 \rightarrow Y_{0,G}$ . Then  $H_i/(H_i \cap G)$  is the inertia group of the point  $\gamma'_i$ . Under our current hypothesis,  $Y_{0,G} \rightarrow X_0 = \mathbb{P}_{\bar{F}}^1$  is a cyclic  $p'$ -cover, totally ramified at the two branch points (hence  $Y_{0,G}$  is a projective  $\bar{F}$ -line). Thus for  $i = 1, 2$ ,  $H_i/(H_i \cap G)$  is cyclic of order  $|A/G|$ .

After descending to  $k$ , the same holds for the subcover  $Y_{1,G} \rightarrow X_1$  of  $\phi_1 : Y_1 \rightarrow X_1$ . That is, there exist  $\delta_1, \delta_2 \in \Delta_{Y_1}$  such that for every  $\delta \in \Delta_{Y_1} - \{\delta_1, \delta_2\}$  the inertia group of any point over  $\delta$  lies in  $G$ . By construction  $Y_{1,G} \rightarrow X_1 = \mathbb{P}_k^1$  is regular, Galois with Galois group  $A/G$ , and branched at only two points. Again, for  $i = 1, 2$ , the inertia group of a point  $\hat{\delta}_i \in Y_1$  lying over  $\delta_i$  is  $A$ -conjugate to  $H_i$ , and the inertia group of its image  $\delta'_i \in Y_{1,G}$  via  $Y_1 \rightarrow Y_{1,G}$  is  $A$ -conjugate to  $H_i/(H_i \cap G)$ . This latter group is cyclic of order  $|A/G|$ . Thus each of  $\delta'_1$  and  $\delta'_2$  is totally ramified and hence is a  $k$ -rational point. Summarizing,  $Y_{1,G} \rightarrow X_1 = \mathbb{P}_k^1$  is regular,  $p'$ -cyclic, branched at only two points, and ramified totally at two  $k$ -rational points. Using Riemann-Hurwitz, we get  $Y_{1,G}$  is rational over  $k$ .

Now consider the intermediate cover  $Z \rightarrow Z_G$  of  $Z \rightarrow W$ . By 4.1.3,  $Z_G = Y_{1,G} \times_k k' = \mathbb{P}_k^1 \times_k k' = \mathbb{P}_{k'}^1$ . Since  $Z_G \rightarrow W$ , the genus of  $W$  must be zero. From 4.1, the cover  $Z \rightarrow W$  is an intermediate cover of a Galois cover  $Z \rightarrow X_1$  with Galois group  $A \times_k A/G$ . Its branch locus is  $\Delta_{Y_1}$  (from  $\phi_1$ ). Its inertia groups are the same as that of  $\phi_1 : Y_1 \rightarrow X_1$ , however now the inertia groups are considered as subgroups of  $A \times_k A/G$  under the inclusion  $\iota$ . For  $i = 1, 2$ , suppose that  $\hat{\tau}_i \in Z$  lies over  $\delta_i \in \Delta_{Y_1} \subset X_1$  and let  $H$  be its inertia group with respect to  $Z \rightarrow X_1$ . Recall that  $\delta_1$  and  $\delta_2$  are the only points whose corresponding inertia groups are not contained in  $\iota(G)$ . Then  $H$  is  $A$ -conjugate to  $H_i$  (actually to  $\iota(H_i)$ ). Thus  $\hat{\tau}_i$  maps to  $\tau'_i \in Z_G$  where  $\tau'_i$  is ramified over  $\delta_i \in X_1$ . The inertia group of  $\tau'_i$  is  $H/(H \cap G)$  which is then  $A$ -conjugate to  $H_i/(H_i \cap G)$ , and this in turn is cyclic of order  $|A/G|$  (as above). Now, since  $G$  is the geometric Galois group of the cover  $Z \rightarrow W$ , the point  $\tau'_i \in Z_G$  is unramified in  $Z_G \rightarrow W$ . Let  $\tau''_i \in W$  be the image of  $\tau'_i$  under  $Z_G \rightarrow W$ . Then the inertia group of  $\tau''_i$  with respect to  $W \rightarrow X_1$  has order equal to that of  $H_i/(H_i \cap G)$ , and this in turn has order  $|A/G|$ . Since the degree of  $W \rightarrow X_1$  is  $|A/G|$ , the point  $\tau''_i$  must be totally ramified over  $\delta_i \in X_1$ .

Since  $\delta_i \in X_1 = \mathbb{P}_k^1$  is a  $k$ -rational point, this implies that  $\delta_i \in W$  is a  $k$ -rational point. As a result,  $W$  is  $k$ -rational.  $\square$

We can restate the previous result in terms of permutation groups.

**COROLLARY 4.4.** *Let  $F$  be any perfect field and let  $A$  be a transitive degree  $n$  permutation group containing a transitive normal subgroup  $G$  and a transitive cyclic-by- $p$  subgroups  $I_1, \dots, I_t$  of  $G$ . Let  $H$  be the stabilizer of a point. Assume that  $A/G \in G(F)$ . Then there exists a cover  $Y \rightarrow X$  defined over some  $k/F$  such that: The arithmetic monodromy group is  $A$ ; the geometric monodromy group is  $G$ ;  $Y$  corresponds to  $H$  in the Galois correspondence; and there exist points  $\hat{\gamma}_i \in Z$  with inertia group  $I_i$ . In particular:*

- (1) *If  $A/G$  is cyclic, then we may take  $F$  a finite field.*
- (2) *If  $A/G$  is cyclic  $p'$ , then we may take  $F$  finite and  $X = \mathbb{P}_F^1$ .*

**PROOF.** Let  $H$  be a point stabilizer in the permutation representation of  $A$ . Since  $G$  is transitive,  $A = GH$ . By the previous result, we can produce a curve  $Z$  defined over a finite extension  $k'$  of  $F$  with  $A$  a group of automorphisms of  $Z$  such that  $G$  is the subgroup fixing  $k'$ . Moreover, we can designate the  $I_j$  to be inertia groups. Let  $k$  denote the fixed field of  $A$  on  $k'$ . Let  $Y = Z_H$  and  $X = Z_A$ . Since  $A = GH$ , the fixed field of  $H$  on  $k'$  is precisely  $k$ . Thus, we have a cover  $f : Y \rightarrow X$  defined over  $k$ . By construction,  $A$  and  $G$  are the arithmetic and geometric monodromy groups of the cover.

If  $A/G$  is cyclic, the previous result also ensures that we may take  $F$  to be finite. In addition, if  $A/G$  is cyclic of order prime to  $p$ , then we may take  $X = \mathbb{P}_F^1$ .  $\square$

The following result relates to Abhyankar's Conjecture for  $\mathbb{A}^1 - \{0\}$  over a finite field.

**THEOREM 4.5.** *Let  $A$  be a finite group generated by  $p(A)$  and one other element  $a \notin p(A)$ . Then there exists a finite extension  $k/\mathbb{F}_p$  and a Galois cover  $\psi : Z \rightarrow \mathbb{P}_k^1$  such that the arithmetic Galois group is  $A$ , the geometric Galois group is  $p(A)$ , and the branch locus consists of only one  $k$ -rational point. Moreover, the resulting regular Galois cover with Galois group  $p(A)$  is a cover of the projective  $k'$ -line for some finite extension  $k'/k$ .*

**PROOF.** Let  $S$  be a  $p$ -Sylow subgroup of  $A$ . By the result of Schur-Zassenhaus, we may assume that  $A = \langle a \rangle p(A)$  with  $\langle a \rangle \subset N_A(S)$ . Let  $X = \mathbb{P}_{\bar{\mathbb{F}}_p}^1$  where  $\bar{\mathbb{F}}_p$  is the algebraic closure of  $\mathbb{F}_p$ . By the Abhyankar Conjecture (new proof in Proposition 4.1 of [Ha99]), we know that  $A$  is the Galois group of a Galois cover  $Y_0 \rightarrow X_0$  such that  $\Delta_{Y_0} = \{\delta_0, \sigma_0\}$  is the branch locus. Moreover, by that result, we may assume that the inertial description of this cover is  $(S \rtimes \langle a \rangle, \langle a^{-1} \rangle)$ . Notice that the subcover  $Y_{0,p(A)} \rightarrow X_0$  induced by  $p(A) \subset A$  is  $p'$ -cyclic, branched at only two points, and is totally ramified over these two points. Thus  $Y_{0,p(A)} \cong \mathbb{P}_{\bar{\mathbb{F}}_p}^1$ .

As in Proposition 4.3, this cover descends to a cover  $Y_1 \rightarrow X_1$  defined over finite extension  $k$  of  $F$  with branch locus  $\{\delta_1, \sigma_1\}$  consisting of  $k$ -rational points and the same inertial description. Moreover,  $X_1 = \mathbb{P}_k^1$ , and we may assume that there is  $k'/k$  Galois with Galois group  $A/p(A)$ . The subcover  $Y_{1,p(A)} \rightarrow X_1$  induced by  $p(A) \subset A$  is  $p'$ -cyclic, branched at only two points, and is totally ramified over these two points (since the inertia groups do not change). These points will also

be  $k$ -rational. Thus  $Y_{1,p(A)}$  has genus zero and a  $k$ -rational point, so  $Y_{1,p(A)}$  is  $k$ -rational.

Applying Proposition 4.1 with  $G = p(A)$  to the descended cover, we obtain a Galois cover  $Z \rightarrow X_1$  dominating  $Y_1 \rightarrow X_1$  with Galois group  $A \times A/p(A)$ . Moreover there is an intermediate cover  $Z \xrightarrow{\psi} W \rightarrow X_1$ . The cover  $\psi$  has arithmetic Galois group  $A$  and geometric Galois group  $p(A)$ . As in Proposition 4.3, we may assume that  $W$  is rational over  $k$  (same argument works, only it is simpler as there are only two branch points for  $Z \rightarrow X_1$ ). Moreover, by 4.1.3, for any branch point  $\beta \in W$ , its inertia group (with respect to  $\psi$ ) is  $A$ -conjugate to  $I \cap p(A)$  where  $I$  is an inertia group of  $Y_1 \rightarrow \mathbb{P}_k^1$ . If  $\beta$  lies over  $\delta_1 \in X_1$  then its inertia group is  $A$ -conjugate to  $(S \rtimes \langle a \rangle) \cap p(A) = S$ . If  $\beta$  lies over  $\sigma_1$  then its inertia group is  $A$ -conjugate to  $\langle a \rangle \cap p(A) = e$ .

Combining all of the above, we have a Galois cover  $\psi : Z \rightarrow W = \mathbb{P}_k^1$  with arithmetic Galois group  $A$  and geometric Galois group is  $G = p(A)$ . The branch locus consists of only one point with inertia group  $S$ . The resulting  $p(A)$ -Galois cover is a cover of the projective  $k'$ -line.  $\square$

REMARK 4.6. As remarked in the introduction, one could also use the proof of Theorem B from [P] to prove the above result. The authors again thank Florian Pop for forwarding to them the argument below. (See Appendix of loc.cit. for the notation.) Let  $K = \kappa((t))$  be the Laurent series field in one variable over some base field  $\kappa$ . Let  $EP = (\gamma, \alpha)$  be a split embedding problem for the fundamental group of some affine curve  $Z$  over  $\kappa$  such that  $C = \ker(\alpha)$  is a quotient of  $\pi_1$  of the affine line over  $\kappa$  and  $C$  has a complement in  $B$  which normalizes a Sylow  $p$  of  $C$ . Then EP has proper solutions for the fundamental group of  $Z$  viewed over  $K$  (see the discussion at the beginning of Proof of Theorem B in loc.cit. on p.569-570, before Movement 1). Suppose that  $\kappa$  is a finite field, and we start with an EP for its absolute Galois group  $G_\kappa$  in which  $C$  is a quasi  $p$  group. We would like to solve EP over the affine line of some (finite) extension  $\lambda$  of  $\kappa$ . Suppose  $C$  is the Galois group of some regular etale cover of the affine line over  $\kappa$ . (We know that every quasi  $p$ -group satisfies this over *some*  $\kappa$ , but probably not over the given  $\kappa$ ). Then for every (finite) extension  $\lambda|\kappa$  of sufficiently large degree the embedding problem EP viewed over  $\lambda$  has proper solutions over  $\lambda$ . (This last part is a standard descent argument.)

## 5. Exceptional Covers and Covers with a totally ramified point

There is an analogy between group theoretic properties and properties of covers. Let  $f : Y \rightarrow X$  be a regular separable branched cover (typically non-Galois) over a field  $k$  of characteristic  $p$  (which we assume to be perfect for simplicity). Consider the Galois closure of  $k(Y)/k(X)$ . This corresponds to a curve  $Z$  defined over a finite Galois extension  $k'/k$ . Let  $A$  be the monodromy group of the cover and let  $G$  be the geometric monodromy group. Let  $H$  be the subgroup of  $A$  corresponding to  $k(Y)$ . Let  $\Omega$  be the  $A$ -set consisting of the left cosets of  $H$ . Then  $A$  and  $G$  act transitively and faithfully on the cosets of  $H$ .

The general problem is to classify covers satisfying certain arithmetic and geometric properties. One approach to this is to translate these arithmetic and geometric properties into properties of the monodromy group of the cover. We then try

to find all group theoretic solutions to the problem. Finally, we need to determine what coverings of curves correspond to the group theoretic solution.

In characteristic zero over an algebraically closed field, using Riemann's existence theorem and the description of fundamental groups of Riemann surfaces minus a finite number of points, one often (but certainly not always) can pass between groups and curves fairly easily. It is much more difficult to do in positive characteristic or when the field is not algebraically closed or both.

We show that the results of the previous section shed some light on some of these problems by showing that for certain problems, every group theoretic solution corresponds to a covering.

The first problem we consider is when there is a totally ramified point  $y \in Y$ . The translation to group theory is rather easy in this case. Let  $I$  be the inertia group of some point over  $y$ . The fact that  $y$  is totally ramified says that after completing at  $y$  (and extending scalars), the local extension has the same degree as the global extension. Thus, the local Galois group acts transitively on the conjugates of a generator. Of course,  $I$  is the local Galois group and so  $I$  is transitive (or equivalently,  $A = HI$ ). Since  $I$  is an inertia group,  $I/O_p(I)$  is cyclic.

By Corollary 4.4, we can produce a cover with the appropriate monodromy groups and with a specified inertia group. Thus, we have the following:

**THEOREM 5.1.** *Let  $A$  be a transitive degree  $n$  permutation group containing a transitive normal subgroup  $G$  with point stabilizer  $H$  and a transitive cyclic-by- $p$  subgroup  $I \leq G$ .*

- (1) *There exists a cover  $f : Y \rightarrow X$  defined over some field  $k$  such that  $A$  is the monodromy group of  $f$ ,  $G$  is the geometric monodromy group of  $f$ ,  $k(Y)$  is the fixed field of  $H$ , and  $I$  is the inertia group of a totally ramified  $k$ -rational point.*
- (2) *If  $A = G$  and  $k$  is algebraically closed, we may take  $f : Y \rightarrow \mathbb{P}^1$  as above to be defined over  $k$ .*
- (3) *If  $A/G$  is cyclic, we may take  $k$  to be a finite field.*
- (4) *If  $A/G$  is a cyclic  $p'$ -group, we may take  $k$  to be a finite field and  $X = \mathbb{P}^1$ .*

**REMARK 5.2.** (1) The results here can be strengthened to prescribe the higher ramification groups by applying a stronger version of Proposition 4.1.

- (2) **Classification:** Recall that a primitive group  $G$  is one that corresponds to a primitive permutation representation. The group  $G$  being primitive is equivalent to the cover being indecomposable (i.e. it does not factor through another non-trivial cover). Assume in addition to the hypotheses of the theorem,  $A$  is a primitive permutation group. Then all such groups have been classified in [GS]. Thus, the theorem above shows that any such group will occur (and moreover, the cover will be indecomposable).

One big open problem is to determine which groups  $A$  occur as monodromy groups in the case that both  $Y$  and  $X$  have genus zero and the cover is indecomposable with a totally ramified rational point. Equivalently, which groups correspond to indecomposable polynomial covers. Our methods do not handle this because we cannot prescribe the genus of  $Y$  in our construction.

In characteristic zero, since all ramification is tame and since we can appeal to Riemann's Existence Theorem, one can classify all such groups

(See [Fe], [Mu] – one still has to invoke the classification of finite simple groups).

If there is wild ramification, the corresponding Riemann-Hurwitz formula is much more difficult to use.

We give one final application. One says that the cover  $f : Y \rightarrow X$  is exceptional over  $k$  if the variety  $\{(u, v) \in Y \times Y \mid f(u) = f(v)\}$  has no absolutely irreducible components defined over  $k$  other than the diagonal. If  $k$  is a sufficiently large finite field, this is equivalent to  $f$  being bijective on  $k$ -rational points. The group theoretic translation of this is that  $A$  and  $G$  have no common orbits on  $\Omega \times \Omega$  other than the diagonal. There are other group theoretic formulations – particularly when  $A/G$  is cyclic. See [FGS] for more details.

Suppose that  $\Omega$  is a finite set and  $A$  is a transitive group of permutations on  $\Omega$ . Assume moreover that  $G$  is a normal subgroup of  $A$  that is also transitive and  $A$  and  $G$  have no common orbits on  $\Omega \times \Omega$  other than the diagonal. We say that  $(A, G, \Omega)$  is exceptional. The earlier results show that any exceptional  $(A, G, \Omega)$  corresponds to an exceptional cover.

The following theorem is now an immediate consequence of the results of Proposition 4.1.

**THEOREM 5.3.** *Suppose that  $(A, G, \Omega)$  is exceptional. Fix a prime  $p$ .*

- (1) *There exists a cover  $f : Y \rightarrow X$  defined over some field  $k$  of characteristic  $p$  such that  $A$  is the monodromy group of  $f$ ,  $G$  is the geometric monodromy group of  $f$ ,  $k(Y)$  is the fixed field of a point stabilizer in  $G$  on  $\Omega$ , and the cover is exceptional.*
- (2) *If  $A/G$  is cyclic, we may take  $k$  to be a finite field.*
- (3) *If  $A/G$  is a cyclic  $p'$ -group, we may take  $k$  to be a finite field and  $X = \mathbb{P}^1$ .*

We close by noting that if we are given  $A, G, \Omega, I$  with  $(A, G, \Omega)$  exceptional and  $I$  a transitive subgroup of  $G$  that is cyclic modulo a normal  $p$ -subgroup, then just as above we can produce a cover that is both exceptional and has a totally ramified rational point.

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