

SEMI-DIRECT GALOIS COVERS OF THE AFFINE LINE

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ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$. Let G be a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ where ℓ is a prime distinct from p . In this paper, we study Galois covers $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group G . We find the minimal genus of a curve Z that admits such a cover and show that it depends only on ℓ , p , and the order a of ℓ modulo p . We also prove that the number of curves Z of this minimal genus which admit such a cover is at most $(p - 1)/a$.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p > 0$. In sharp contrast with the situation in characteristic 0, there exist Galois covers $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over infinity. By Abhyankar's Conjecture [1], proved by Raynaud and Harbater [7], [3], a finite group G occurs as the Galois group of such a cover ψ if and only if G is quasi- p , i.e., G is generated by p -groups. This result classifies all the finite quotients of the fundamental group $\pi_1(\mathbb{A}_k^1)$. It does not, however, determine the profinite group structure of $\pi_1(\mathbb{A}_k^1)$ because this fundamental group is an infinitely generated profinite group.

There are many open questions about Galois covers $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over infinity. For example, given a finite quasi- p group G , what is the smallest integer g for which there exists a cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over infinity with Z of genus g ? As another example, suppose G and H are two finite quasi- p groups such that H is a quotient of G . Given an unramified Galois cover ϕ of \mathbb{A}_k^1 with group H , under what situations can one dominate ϕ with an unramified Galois cover ψ of \mathbb{A}_k^1 with Galois group G ? Answering these questions will give partial progress towards understanding how the finite quotients of $\pi_1(\mathbb{A}_k^1)$ fit together in an inverse system. These questions

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are more tractable for quasi- p groups that are p -groups since the maximal pro- p quotient $\pi_1^p(\mathbb{A}_k^1)$ is free (of infinite rank) [8].

In this paper, we study Galois covers $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$, where ℓ is a prime distinct from p . Such a cover ψ must factor through an Artin-Schreier cover $\phi : Y \rightarrow \mathbb{P}_k^1$, which has an affine equation $y^p - y = f(x)$ for some $f(x) \in k[x]$ with degree s prime-to- p . The ℓ -torsion $\text{Jac}(Y)[\ell]$ of the Jacobian of Y is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^{2gy}$. When $f(x) = x^s$, we determine how an automorphism τ of Y of order p acts on $\text{Jac}(Y)[\ell]$. This allows us to construct a Galois cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ which dominates ϕ , such that the Galois group of ψ is $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ where a is the order of ℓ modulo p (Section 3). We also determine the smallest integer g that can occur as the genus of Z for a Galois cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ and find an upper bound on the number of curves Z of that genus which admit such a cover (Section 4).

2. QUASI- p SEMI-DIRECT PRODUCTS

We describe semi-direct products of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ which can occur as Galois groups for covers of \mathbb{P}_k^1 ramified only over ∞ .

Definition 2.1. *A finite group is a quasi p -group if it is generated by all of its Sylow p -subgroups.*

It is well-known that there are other equivalent formulations of the quasi- p property, such as the next result.

Lemma 2.2. *A finite group is a quasi p -group if and only if it has no nontrivial quotient group whose order is relatively prime to p .*

The importance of the quasi- p property is that it characterizes which finite groups occur as Galois groups of unramified covers of the affine line.

Theorem 2.3. *A finite group occurs as the Galois group of a Galois cover of the projective line \mathbb{P}_k^1 ramified only over infinity if and only if it is a quasi- p group.*

Proof. This is a special case of Abhyankar's Conjecture [1] which was jointly proved by Harbater [3] and Raynaud [7]. \square

We now restrict our attention to groups of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$.

Lemma 2.4. *Suppose a quasi- p group G is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$.*

- (1) *Then G is not a direct product.*
- (2) *Moreover, $b \geq \text{ord}_p(\ell)$ where $\text{ord}_p(\ell)$ is the order of ℓ modulo p .*

Proof. Part (1) is true since $(\mathbb{Z}/\ell\mathbb{Z})^b$ cannot be a quotient of the quasi- p group G .

For part (2), the structure of a semi-direct product $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ depends on a homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/\ell\mathbb{Z})^b$. By part (1), ι is an inclusion. Thus $\text{Aut}(\mathbb{Z}/\ell\mathbb{Z})^b \simeq \text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})$ has an element of order p . Now

$$|\text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})| = (\ell^b - 1)(\ell^b - \ell) \cdots (\ell^b - \ell^{b-1}).$$

Thus $\ell^\beta \equiv 1 \pmod{p}$ for some $\beta \leq b$ which implies $b \geq \text{ord}_p(\ell)$. \square

Lemma 2.5. *If $a = \text{ord}_p(\ell)$, then there exists a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ which is quasi- p .*

Proof. If $a = \text{ord}_p(\ell)$, then there is an injective homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^a)$. Thus there exists a non-abelian semi-direct product G of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. To show that G is quasi- p , suppose N is a normal subgroup of G whose index is relatively prime to p . Then N contains an element τ of order p . Since G is non-abelian, it contains no normal subgroup of order p and so N also contains an element of order ℓ . Recall that $\text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^\beta)$ contains no element of order p for $\beta < a$. Thus τ acts irreducibly on the elementary abelian subgroup $(\mathbb{Z}/\ell\mathbb{Z})^a$ of G . Thus $N = G$ and G has no non-trivial quotient group whose order is relatively prime to p . By Lemma 2.2, G is quasi- p . \square

3. EXISTENCE OF $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ -GALOIS COVERS OF \mathbb{A}_k^1

In this section, we give concrete examples of Galois covers $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. To determine the genus of the covering curve Z , we will need to determine the higher ramification groups of ψ .

Definition 3.1. *Let L/K be a Galois extension of function fields with Galois group G and let P, P' be primes of K and L such that $P'|P$. Let $\nu_{P'}$ and $\mathcal{O}_{P'}$ be the corresponding valuation function and valuation ring for P' . For any integer $i \geq -1$ we define the i th ramification group of $P'|P$ as*

$$I_i(P'|P) = \{\sigma \in G \mid \nu_{P'}(\sigma(z) - z) \geq i + 1, \forall z \in \mathcal{O}_{P'}\}.$$

Lemma 3.2. *Suppose $f(x) \in k[x]$ is a polynomial of degree s for a positive integer s prime to p . Let $\phi : Y \rightarrow \mathbb{P}_k^1$ be the cover of curves corresponding to the field extension*

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)).$$

- (1) *Then $\phi : Y \rightarrow \mathbb{P}_k^1$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point P_∞ over ∞ .*
- (2) *The i th ramification group at P_∞ satisfies*

$$I_i = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \leq s \\ 0 & \text{if } i > s. \end{cases}$$

- (3) *The genus g_Y of Y is equal to*

$$g_Y = (p - 1)(s - 1)/2.$$

Proof. For part (1), note that the extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x))$ is cyclic of degree p , with Galois group generated by the automorphism $\tau : y \mapsto y + 1$ of order p . Let P be a finite prime of $k(x)$ and let ν_P be the corresponding valuation. Then $\nu_P(f(x)) \geq 0$, hence P is unramified by [10, Prop. III.7.8(b)]. For the infinite prime ∞ with corresponding valuation ν_∞ , we have

$$\nu_\infty(f(x) - (z^p - z)) \leq 0$$

for all $z \in k[x]$ thus P_∞ is totally ramified by [10, Prop. III.7.8(c)].

To prove part (2), we note that furthermore

$$\nu_{P_\infty}(y^p - y) = \nu_{P_\infty}(f(x)) = \nu_{P_\infty}(x^s) = -sp,$$

which implies that

$$\nu_{P_\infty}(y) = -s.$$

Now let $\widehat{\theta}$ be the completion of the valuation ring of $k(x)[y]/(y^p - y - f(x))$ at P_∞ , and let π_∞ be a generator of the unique prime in $\widehat{\theta}$. Then, write $y = \pi_\infty^{-s}u$, where u is a unit in $\widehat{\theta}$. Since $\sqrt[s]{y}$ is an element of $\widehat{\theta} \simeq k[[\pi_\infty]]$, we can assume without loss of generality that $\sqrt[s]{y} = \pi_\infty^{-1}$. Recalling that τ acts on y by $\tau(y) = y + 1$, we have

$$\begin{aligned} \tau(\pi_\infty) &= \tau(1/y)^{1/s} = (\pi_\infty^s/(1 + \pi_\infty^s))^{1/s} \\ &= \pi_\infty(1 - \pi_\infty + \pi_\infty^{2s} - \dots)^{\frac{1}{s}} \\ &= \pi_\infty - a_{s+1}\pi_\infty^{s+1} + a_{2s+1}\pi_\infty^{2s+1} - \dots \end{aligned}$$

for some non-zero coefficient a_{s+1} . Thus $\nu_{P_\infty}(\tau(\pi_\infty) - \pi_\infty) = s + 1$, which completes the proof of part (2).

To find the genus g_Y of Y for part (3), we make use of the Riemann-Hurwitz formula

$$2g_Y - 2 = p(-2) + \sum_{i=0}^{\infty} (|I_i| - 1),$$

where I_i denotes the i th ramification group at P_∞ , [4, Thms. 7.27 & 11.72]). From part (2), we then obtain that $g_Y = (p - 1)(s - 1)/2$. \square

We shall soon prove a general result about $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ -covers of \mathbb{P}_k^1 ramified only over ∞ . But before we do so, we start with a specific example.

Example 3.3. Let p be an odd prime. Consider the Artin-Schreier cover $\phi : Y_2 \rightarrow \mathbb{P}_k^1$ corresponding to the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^2)$. By Lemma 3.2(3), the genus of Y_2 is $g_Y = (p - 1)/2$.

Let $\text{Jac}(Y)$ be the Jacobian of Y . The automorphism τ of Y given by $\tau(y) = y + 1$ defines an automorphism of $\text{Jac}(Y)$ of order p .

Now we describe the action of τ on the subgroup $\text{Jac}(Y)[2]$ of 2-torsion points of $\text{Jac}(Y)$ explicitly. Note that since $2g_Y = (p - 1)$, then $\text{Jac}(Y)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ by [6, pg. 64]. Thus we can represent τ as an element of $\text{GL}_{p-1}(\mathbb{Z}/2\mathbb{Z})$.

For $0 \leq i \leq p-1$, the point $P_i = (0, i)$ can be viewed as a divisor and $D_i = P_i - P_\infty$ can be viewed as an element of $\text{Jac}(Y)$. Then since $O = \text{div}(y - i) = 2D_i$ and $O = \text{div}(x) = D_0 + D_1 + \cdots + D_{p-1}$, we have $D_i \in \text{Jac}(Y)[2]$ with the only relation $D_{p-1} = -(D_0 + D_1 + \cdots + D_{p-2})$. In particular, D_0, \dots, D_{p-2} form a basis of $\text{Jac}(Y)[2]$. With respect to this basis, the action of τ can be represented by the $(p-1) \times (p-1)$ -matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & 0 & -1 \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial of τ is the p th cyclotomic polynomial $\Phi_p(t) = 1 + t + \cdots + t^{p-1} \in (\mathbb{Z}/2\mathbb{Z})[t]$, which factors into irreducible polynomials each of degree equaling the order of 2 modulo p . In particular, τ acts irreducibly on $\text{Jac}(Y)[2]$ if and only if 2 is a primitive root modulo p , i.e., if and only if p is an Artin prime.

For example, if $p = 3$, then τ acts irreducibly on $\text{Jac}(Y)[2]$ with minimal polynomial $\Phi_3(t) = t^2 + t + 1$ and this action can be represented by the 2×2 -matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

If $p = 7$, then 2 has order 3 modulo 7 and the factorization of $\Phi_7(t)$ into irreducible polynomials is $\Phi_7(t) \equiv (x^3 + x^2 + 1)(x^3 + x + 1)$ modulo 2. Thus the action of τ on $\text{Jac}(Y)[2]$ can be represented by the 6×6 -matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 and A_2 are the irreducible 3-dimensional companion matrices of $x^3 + x^2 + 1$ and $x^3 + x + 1$ respectively.

For the rest of the paper, let $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$ be the Artin-Schreier cover corresponding to the field extension

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s).$$

We show that ϕ_s can be dominated by a Galois cover of \mathbb{P}_k^1 with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ for a equal to the order of ℓ modulo p .

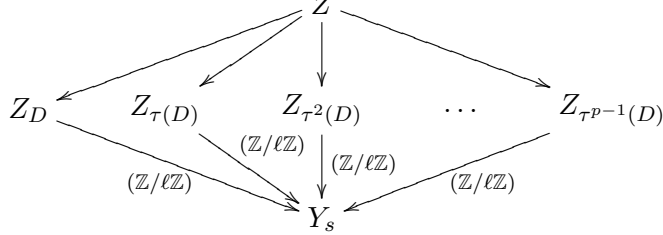
Proposition 3.4. *Let s and ℓ be primes distinct from p . Let $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$ be the Artin-Schreier cover with affine equation $y^p - y = x^s$. Let $a = \text{ord}_p(\ell)$ be the order of ℓ modulo p . Then there exists an unramified Galois cover $\omega : Z \rightarrow Y_s$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a$ such that $\phi_s \circ \omega : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover of \mathbb{P}_k^1 ramified only over ∞ with Galois group a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$.*

Proof. By Lemma 3.2(1), $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point P_∞ over ∞ . The genus g_s of Y_s is $(p-1)(s-1)/2$. Consider two automorphisms of Y_s defined by

$$\tau : \begin{cases} x \mapsto x, \\ y \mapsto y + 1, \end{cases} \quad \sigma : \begin{cases} x \mapsto \zeta_s x, \text{ where } \zeta_s \text{ is a primitive } s\text{th root of unity,} \\ y \mapsto y. \end{cases}$$

Let $\text{Jac}(Y_s)$ be the Jacobian of Y_s . Then τ and σ define automorphisms of $\text{Jac}(Y_s)$ of orders p and s respectively. Therefore, $\text{End}(\text{Jac}(Y_s))$ contains a ring isomorphic to $\mathbb{Z}[\zeta_p, \zeta_s] \cong \mathbb{Z}[\zeta_{ps}]$, which is a \mathbb{Z} -module of rank $\phi(ps) = (p-1)(s-1) = 2g_s$. Then $\mathbb{Q}(\zeta_{ps})$ is contained in $\text{End}(\text{Jac}(Y_s)) \otimes \mathbb{Q}$. In other words, $\text{Jac}(Y_s)$ has complex multiplication by $\mathbb{Q}(\zeta_{ps})$.

For a prime ℓ distinct from p , the automorphism τ induces an action on the subgroup $\text{Jac}(Y_s)[\ell]$ of ℓ -torsion points of $\text{Jac}(Y_s)$. Recall that there is a bijection between ℓ -torsion points D of $\text{Jac}(Y_s)$ and unramified $(\mathbb{Z}/\ell\mathbb{Z})$ -Galois covers $\omega_D : Z_D \rightarrow Y_s$ [5, Prop. 4.11]. Also D has order ℓ if and only if Z_D is connected. This induces a bijection between orbits of τ on the set of unramified $(\mathbb{Z}/\ell\mathbb{Z})$ -Galois covers $\omega_D : Z_D \rightarrow Y_s$ and on the set of ℓ -torsion points of $\text{Jac}(Y_s)$. For a point D of order ℓ of $\text{Jac}(Y_s)$, consider the compositum $\omega : Z \rightarrow Y_s$ of all of the conjugates $\omega_{\tau^j(D)} : Z_{\tau^j(D)} \rightarrow Y_s$ for $0 \leq j \leq p-1$:



Then Z is invariant under τ and so $\phi_s \circ \omega : Z \rightarrow \mathbb{P}_k^1$ is Galois. Moreover, $\phi_s \circ \omega$ is the Galois closure of $\phi_s \circ \omega_D : Z_D \rightarrow \mathbb{P}_k^1$.

Suppose there is a non-trivial one-dimensional τ -invariant subspace of $\text{Jac}(Y_s)[\ell]$ with eigenvalue 1; i.e. τ acts trivially on this subgroup of order ℓ . This yields a cover $\psi_s \circ \omega_1 : Z_1 \rightarrow Y_s \rightarrow \mathbb{P}_k^1$. Since the action of τ is trivial, $\psi_s \circ \omega_1$ is Galois, ramified only over ∞ , with abelian Galois group $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. This contradicts Lemma 2.4(1).

Since τ has order p , the minimal polynomial $m_\tau(t)$ of τ divides $t^p - 1 = (t-1)(t^{p-1} + \dots + 1)$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$. From the preceding paragraph, there is no non-trivial one-dimensional τ -invariant subspace of $\text{Jac}(Y_s)[\ell]$ with eigenvalue 1. This implies that $m_\tau(t)$ divides the p th cyclotomic polynomial $\Phi_p(t) = t^{p-1} + \dots + 1$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$.

Recall that $\Phi_p(t)$ is the minimal polynomial of a primitive p th root of unity ζ_p . Now $\mathbb{Q}(\zeta_p)$ is a Galois extension of \mathbb{Q} , unramified over ℓ since $\ell \neq p$, and all primes over ℓ have the same residue field degree. The irreducible factors of $\Phi_p(t)$ modulo ℓ are in one-to-one correspondence with the primes

of $\mathbb{Z}[\zeta_p]$ over ℓ , and each of their degrees is equal to the residue field degree of the corresponding prime over ℓ . The latter equals the order $a = \text{ord}_p(\ell)$ of ℓ modulo p [2, Ch. 12.2, Exercise #20]. Thus the degree of the minimal polynomial $m_\tau(t)$ of τ equals a .

Since $2g_s = (p-1)(s-1)$, we have $\text{Jac}(Y_s)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{(p-1)(s-1)}$, so we can represent τ as an element of $\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$. We can choose a basis of $\text{Jac}(Y_s)[\ell]$ such that τ is represented as an element of $\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$ in block form. The first irreducible subrepresentation of τ has dimension a . Moreover, since $\mathbb{Q}(\zeta_{ps})$ is a Galois extension of \mathbb{Q} , the block form of τ consists entirely of irreducible blocks of the same size. In particular, the number of irreducible blocks is $(p-1)(s-1)/a$. In other words, τ can be represented by an element of $\text{GL}_{(s-1)(p-1)}(\mathbb{Z}/\ell\mathbb{Z})$ of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{(p-1)(s-1)/a} \end{pmatrix},$$

where A_i is an $a \times a$ matrix representing an a -dimensional irreducible subrepresentation of τ on $\text{Jac}(Y_s)[\ell]$.

Using the bijection between orbits of $\text{Jac}(Y_s)[\ell]$ and orbits of $(\mathbb{Z}/\ell\mathbb{Z})$ -covers of Y_s under τ and the above observation for the action of τ on $\text{Jac}(Y_s)[\ell]$, there exists an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover $\omega : Z \rightarrow Y_s$ such that $\phi_s \circ \omega : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover of \mathbb{P}_k^1 with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. Also $\phi_s \circ \omega$ is ramified only over infinity since ϕ_s is ramified only over ∞ and since ω is unramified. \square

4. MINIMAL GENUS OF $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ -GALOIS COVERS OF \mathbb{A}_k^1

In this section, we find the minimal genus of a curve Z that admits a Galois cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ , with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ where a is the order of ℓ modulo p . The minimal genus depends only on ℓ , p , and a . We also prove that the number of curves Z of this minimal genus which admit such a cover is at most $(p-1)/a$.

Theorem 4.1. *Let p be an odd prime. Let ℓ be a prime distinct from p and let a be the order of ℓ modulo p . Then:*

- (1) *There exists a Galois cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$.*
- (2) *The minimal genus of a curve Z which admits a Galois cover as in part (1) is $g_Z = 1 + \ell^a(p-3)/2$.*
- (3) *There are at most $(p-1)/a$ isomorphism classes of curves Z which admit a Galois cover as in part (1) with minimal genus g_Z as in part (2).*

Proof. For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ which is quasi- p . The result is then immediate from Theorem 2.3.

For part (2), we first compute the genus of the curve Z for the Galois cover found in Proposition 3.4. Recall that $\phi_s \circ \omega : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover of \mathbb{P}_k^1 ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. Also $\omega : Z \rightarrow Y_s$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover and Y has genus $g_s = (p-1)/2$. By the Riemann-Hurwitz formula, $2g_Z - 2 = \ell^a(2g_Y - 2) = \ell^a(p-3)$, i.e., $g_Z = 1 + \ell^a(p-3)/2$.

Now we show that this is the smallest genus that can occur for such a cover. Suppose $\psi : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. Since $(\mathbb{Z}/\ell\mathbb{Z})^a$ is normal in $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$, the cover ψ must factor through a Galois cover $\phi : Y \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group $\mathbb{Z}/p\mathbb{Z}$. By Artin-Schreier theory, ϕ is given by an equation $y^p - y = f(x)$ where $f \in k[x]$ has degree s for some integer s relatively prime to p . By the Riemann-Hurwitz formula, $2g_Z - 2 = \ell^a(2g_Y - 2) + R$, where R denotes the contribution from ramification. We see that in order to minimize the genus of Z , it is necessary to take $\omega : Z \rightarrow Y$ unramified and Y of minimum genus among the Artin-Schreier covers of the affine line. Since the genus g_Y of Y is $(p-1)(s-1)/2$ by Lemma 3.2 (3), we should make s as small as possible. The value $s = 1$ is impossible since then Y is a projective line and there do not exist Galois covers of the projective line ramified only over one point with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Thus $s = 2$ yields the smallest possible value for g_Y and thus the quantity $g_Z = 1 + \ell^a(p-3)/2$ is the minimal value of g_Z .

For part (3), suppose $\psi : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ and $g_Z = 1 + \ell^a(p-3)/2$. As in part (2), ψ factors as $\phi \circ \omega$ where $\omega : Z \rightarrow Y$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover, where $\phi : Y \rightarrow \mathbb{P}_k^1$ is an Artin-Schreier cover ramified only over ∞ , and where Y has genus $(p-1)/2$. By Lemma 3.2(3), Y has an affine equation $y^p - y = a_2x^2 + a_1x + a_0$ for some $a_0, a_1 \in k$ and $a_2 \in k^*$. Since p is odd, by completing the square, we can write $a_2x^2 + a_1x + a_0 = x_1^2 + \epsilon$ where $x_1 = \sqrt{a_2}x + a_1/2\sqrt{a_2}$. After modifying by an automorphism of the projective line, specifically by the linear fractional transformation $x \mapsto x_1$, the equation for Y can be rewritten as $y^p - y = x_1^2 + \epsilon$. Since k is algebraically closed, there exists $\delta \in k$ such that $\delta^p - \delta = \epsilon$. Let $y_1 = y - \delta$. After the change of variables $y \mapsto y_1$, the curve Y is isomorphic to the curve Y_2 with affine equation $y_1^p - y_1 = x_1^2$. Thus there is a unique possibility for the isomorphism class of the curve Y .

From the proof of Proposition 3.4, there is a bijection between τ -invariant connected unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois covers of Y_2 and orbits of τ on points D of order ℓ on $\text{Jac}(Y_2)$. The action of τ on $\text{Jac}(Y_2)[\ell]$ decomposes into $(p-1)/a$ irreducible subrepresentations. Each of these is distinct, because the irreducible factors of $\Phi_p(t) \in (\mathbb{Z}/\ell\mathbb{Z})[t]$ are distinct. Thus there are $(p-1)/a$ choices for a τ -invariant unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover of Y_2 .

Thus there are at most $(p-1)/a$ isomorphism classes of curves Z which admit a Galois cover as in part (1) with minimal genus g_Z as in part (2). \square

Theorem 4.2. *Let $p = 2$ and let ℓ be an odd prime. Then:*

- (1) *There exists a Galois cover $\psi : Z \rightarrow \mathbb{P}_k^1$ ramified only over ∞ with Galois group of the form $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.*
- (2) *The minimal genus of a curve Z which admits a Galois cover as in part (1) is $g_Z = 1$.*
- (3) *There are at most $\ell+1$ isomorphism classes of curves Z which admit a Galois cover as in part (1) with minimal genus $g_Z = 1$.*

Proof. Note that the order of ℓ modulo 2 is $a = 1$. For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ which is quasi-2. The result is then immediate from Theorem 2.3.

Suppose $\psi : Z \rightarrow \mathbb{P}_k^1$ is a Galois cover ramified only over ∞ with Galois group as in part (1). As in Theorem 4.1, ψ must factor through an Artin-Schreier cover $\phi : Y \rightarrow \mathbb{P}_k^1$ with affine equation $y^2 - y = f(x)$ for some $f(x) \in k[x]$ of odd degree s . The minimal genus for Z will again occur when $\omega : Z \rightarrow Y$ is unramified and when s is as small as possible. As before, $s = 1$ is impossible, and so $s = 3$ is the smallest choice. In this case, by Lemma 3.2(3), $g_Y = 1$, i.e. Y is an elliptic curve. By the Riemann-Hurwitz formula, the minimal genus for Z is $g_Z = 1 + \ell(g_Y - 1) = 1$, which completes part (2).

For part (3), we can complete the cube of $f(x)$ and make the corresponding change of variables, which is a scaling and translation of x . So we can assume that Y has affine equation $y^2 - y = x^3 + a_1x + a_0$ for some $a_0, a_1 \in k$. Then it follows from [9, Appendix A, Prop.1.1c] that the j -invariant of Y is $j(Y) = 0$ and that the discriminant is $\Delta(Y) = (-1)^4 = 1$. Since k is algebraically closed, by [9, Appendix A, Prop.1.2b], all elliptic curves Y with $j(Y) = 0$ are isomorphic over k . Thus there is a unique choice for Y up to isomorphism. Without loss of generality, we may assume that $Y = Y_3$ has affine equation $y^2 - y = x^3$.

From the proof of Proposition 3.4, the action of τ on $\text{Jac}(Y_3)[\ell]$ decomposes into the direct sum of two 1-dimensional subrepresentations. In other words, the action of τ is diagonal with both eigenvalues equal to -1 . The number of non-trivial τ -invariant subgroups of $\text{Jac}(Y_3)[\ell]$ is the number of subgroups of order ℓ in $(\mathbb{Z}/\ell\mathbb{Z})^2$, which is $\ell + 1$. As in Theorem 4.1, this implies that there are at most $\ell + 1$ isomorphism classes of curves Z which admit a Galois cover as in part (1) with minimal genus $g_Z = 1$. \square

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