

# Abhyankar's Local Conjecture on Fundamental Groups

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**Abstract:** This paper proves the remaining open case of Abhyankar's higher dimensional conjecture on local fundamental groups in characteristic  $p$  ([Ab2], [Ab3]). This conjecture, which is analogous to Abhyankar's conjectures on global fundamental groups, proposed that a finite group  $G$  is a Galois group over  $k[[x_1, \dots, x_n]][(x_1 \cdots x_r)^{-1}]$  if and only if its maximal prime-to- $p$  quotient is, provided  $n \geq 2$  and  $1 \leq r \leq n$ . For  $r > 1$ , this conjecture was disproven in [HP]. Here we prove that the conjecture is true in the case  $r = 1$ . So the Galois groups over  $k[[x_1, \dots, x_n]][x_1^{-1}]$  are precisely the cyclic-by-quasi- $p$  groups.

## Section 1. Introduction.

In 1957, Abhyankar made a conjecture [Ab1] concerning the fundamental group of an affine curve  $X$  over an algebraically closed field  $k$  of characteristic  $p$ . Specifically, his conjecture stated what the finite quotients  $G$  of  $\pi_1(X)$  are — or equivalently, which finite groups are Galois groups of finite unramified connected covers of  $X$ . Namely, if  $X$  is obtained by deleting  $r$  points from a smooth projective curve of genus  $g \geq 0$ , then  $G$  is such a Galois group if and only if its maximal prime-to- $p$  quotient  $G/p(G)$  can be generated by a set of at most  $2g + r - 1$  elements. (Here  $p(G)$  is the subgroup of  $G$  generated by the Sylow  $p$ -subgroups of  $G$ .) As Grothendieck later showed [Gr2], a *prime-to- $p$*  group is a Galois group over  $X$  if and only if it has such a set of generators. Thus Abhyankar's curve conjecture was equivalent to the assertion that a finite group  $G$  is a Galois group over  $X$  if and only if  $G/p(G)$  is. This was proven in the case that  $X = \mathbb{A}^1$  in [Ra], and was then proven for general affine curves in [Ha1].

Generalizing the statement of this conjecture, Abhyankar has proposed that the same principle should govern Galois groups of affine  $k$ -varieties in higher dimensions, in both local and global situations. (This was stated implicitly in [Ab2] and explicitly in [Ab3].) In the global case, he considered the fundamental group of an affine variety  $X$  that is the complement of a normal crossing divisor  $D$  in  $\mathbb{P}^n$ . Say  $D$  has irreducible components  $D_1, \dots, D_r$  of degrees  $d_1, \dots, d_r$ . Then Abhyankar's global conjecture says that a finite group  $G$  is a Galois group over  $X$  if and only if  $G/p(G)$  is an abelian group that is generated by elements  $g_1, \dots, g_r$  satisfying  $g_1^{d_1} \cdots g_r^{d_r} = 1$ . For a *prime-to- $p$*  group,  $G$  is indeed a Galois group over  $X$  if and only if it has such a set of generators [Ab2], [F]. (The corresponding case over  $\mathbb{C}$  had previously appeared in work of Zariski [Z1], [Z2], [F].) Thus

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Abhyankar's higher dimensional global conjecture was again equivalent to saying that a finite group  $G$  is a Galois group over  $X$  if and only if  $G/p(G)$  is.

Similarly, in the local case, Abhyankar proposed his

**Local conjecture.** (Abhyankar) *Let  $n > 1$  and  $1 \leq r \leq n$ . Then a finite group  $G$  is the Galois group of a finite unramified extension of  $\hat{R}_{n,r} := k[[x_1, \dots, x_n]][(x_1 \cdots x_r)^{-1}]$  if and only if its maximal prime-to- $p$  quotient  $G/p(G)$  is abelian and has a generating set of at most  $r$  elements.*

(Here, as elsewhere in this paper, it is understood that the extension is required to be a domain, or equivalently that its spectrum is (reduced and) irreducible.)

A *prime-to- $p$*  group is a Galois group over  $R_{n,r}$  if and only if it is abelian with such a generating set (because of Abhyankar's Lemma; see [Ab2], [HP, Prop. 3.1]). So as before, the conjecture says that a finite group  $G$  is a Galois group over  $X = \text{Spec } \hat{R}_{n,r}$  if and only if  $G/p(G)$  is.

Of course, for  $G$  to be a Galois group over any space  $X$  it is necessary for  $G/p(G)$  to be a Galois group over  $X$ . So in these conjectures, the issue is whether this condition is also sufficient. In [HP], it was shown by the first author and by M. van der Put that this condition is *not* sufficient in either the local and global cases in dimension  $> 1$ . Namely, if  $1 \leq r \leq n$ , and if  $\hat{X}_{n,r} = \text{Spec } \hat{R}_{n,r}$ , then the short exact sequence

$$1 \rightarrow p(\pi_1(\hat{X}_{n,r})) \rightarrow \pi_1(\hat{X}_{n,r}) \rightarrow \pi_1(\hat{X}_{n,r})/p(\pi_1(\hat{X}_{n,r})) \rightarrow 1$$

has a splitting [HP, Cor. 3.4(a)]; and from this it follows that an analogous splitting exists for the fundamental group of  $X_{n,r} := \mathbb{A}^n - (x_1 \cdots x_r = 0)$  [HP, Cor. 4.7(a)]. This splitting imposes an additional non-trivial condition on Galois groups over  $X$  and  $\hat{X}$ , if  $1 < r \leq n$ . So the local and global conjectures do not hold in those cases [HP, Examples 5.2, 5.3], although several possible variants are suggested by [HP, §5].

This leaves open the question of whether Abhyankar's higher dimensional conjectures hold for  $X_{n,1}$  and for  $\hat{X}_{n,1}$ . In these situations, the cokernel of the above exact sequence is free of rank 1 (as a pro-prime-to- $p$  group), hence it automatically splits. So in these two cases, the splitting condition does not impose any new restrictions for a finite group to be a Galois group. And in the case of  $X_{n,1}$ , it is easy to see from Abhyankar's original conjecture for  $\mathbb{A}^1$  that the higher dimensional global conjecture does in fact hold here [HP, Ex. 5.3]. The case of  $\hat{X}_{n,1}$  has remained open, though; and the purpose of the present paper is to prove that Abhyankar's higher dimensional local conjecture does hold for this space:

**Main Theorem.** *For  $n > 1$ , a finite group  $G$  is the Galois group of an unramified extension of  $\hat{R}_{n,1} = k[[x_1, \dots, x_n]][x_1^{-1}]$  if and only if  $G/p(G)$  is cyclic.*

Note that the case  $n = 1$  clearly does not hold, since only cyclic-by- $p$  groups can occur over the Laurent series field  $k((x)) = k[[x]][x^{-1}]$  (using that  $k$  is algebraically closed). Of course this case was not part of Abhyankar's conjectures.

As in [Ha1], the proof will rely on Abhyankar's original conjecture for the affine line [Ra], together with results about embedding problems and methods of formal patching. Since the formal patching methods apply to global objects, not to local ones, we will first use blowings-up in order to introduce exceptional divisors and thereby pass from a local situation to a global one.

Section 2 of this paper contains some related results about Galois covers in a global situation. These are combined with a blowing-up construction in Section 3 in order to prove our Main Theorem above (rephrased as Theorem 3.3 below), saying that Abhyankar's local conjecture holds if (and only if)  $r = 1$ . We provide another viewpoint on this result in Section 4.

Throughout this paper, if  $X$  is a connected scheme, then a *cover*  $f : Y \rightarrow X$  will be a morphism that is finite and generically separable. A *Galois cover*  $f : Y \rightarrow X$  is a connected cover whose covering group  $\text{Aut}_X(Y)$  acts simply transitively on each geometric generic fibre of  $f$ . If  $G$  is a finite group, then a  *$G$ -Galois cover* is a (possibly disconnected) cover  $f : Y \rightarrow X$  together with a homomorphism  $G \rightarrow \text{Aut}_X(Y)$  with respect to which  $G$  acts simply transitively on each generic geometric fibre.

## Section 2. Global results.

This section contains an extension of a result in [Ha3] related to embedding problems over curves in characteristic  $p$ . Recall that a finite group  $Q$  is a *quasi- $p$  group* if  $Q = p(Q)$ . The following proposition is a special case of [Ha3, Prop. 2.3]:

**Proposition 2.1.** *Let  $\Gamma = Q \rtimes G$  be the semi-direct product of a quasi- $p$  group  $Q$  with a finite group  $G$  such that  $G$  normalizes a Sylow  $p$ -subgroup  $P$  of  $Q$ . Let  $Y \rightarrow X$  be a  $G$ -Galois cover of smooth connected projective  $k$ -curves, and let  $\xi_0 \in X$ . Then there is a normal absolutely irreducible  $Q$ -Galois cover  $Z_t \rightarrow Y_t := Y \times_k k((t))$  that is étale away from the fibre over  $\xi_0$ , such that  $Z_t \rightarrow X_t := X \times_k k((t))$  is  $\Gamma$ -Galois.*

In fact, as we show below, even more is true:

**Proposition 2.2.** *In Proposition 1, let  $\delta_1, \dots, \delta_m \in X - \{\xi_0\}$  be distinct non-branch points of  $Y \rightarrow X$ . For  $i = 1, \dots, m$  let  $W_i \rightarrow S$  be (possibly disconnected) smooth  $Q$ -Galois covers of the projective  $s$ -line  $S$ , branched only at  $s = 0$ , where they have  $p$ -group inertia. Let  $\hat{X}_t$  be the blow-up of  $X \times_k k[[t]]$  at the points  $\delta_1, \dots, \delta_m$  on the closed fibre, and let  $\hat{Z}_t$  be the normalization of  $\hat{X}_t$  in  $Z_t$ . Then the cover  $Z_t$  in Proposition 1 may be chosen so that the fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  over the exceptional divisor at  $\delta_i$  is a disjoint union*

of copies of  $W_i \rightarrow S$  away from its branch point, and such that fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  over the proper transform of  $(t = 0)$  is connected.

Here we prove these two results together, essentially following the construction in the proof of [Ha3, Prop. 2.3]. As in that paper, there are three main ingredients: Abhyankar's Conjecture over the affine line [Ra], formal patching [HS], and the existence of solutions to  $p$ -embedding problems in characteristic  $p$  [Ha2].

*Proof of Propositions 2.1 and 2.2.* After adding an additional  $\delta_i$  if necessary and renumbering, we may assume that  $W_1$  is the trivial (disconnected)  $Q$ -Galois cover of the projective  $s$ -line  $S$ . Let  $\tilde{G} = P \rtimes G$  be the subgroup of  $\Gamma$  generated by  $P$  and  $G$ . Let  $\mathcal{K}_i$  be the local field of a point of  $W_i \rightarrow S$  over  $(s = 0)$  whose inertia group  $P_i$  is contained in  $P$ . Meanwhile, by [Ra], there is a smooth connected  $Q$ -Galois cover  $W \rightarrow S$  branched only over  $(s = 0)$ , where its inertia groups are the Sylow  $p$ -subgroups of  $Q$ . Let  $\mathcal{K}$  be the local field of a ramification point of  $W \rightarrow S$  with inertia group  $P$ ; this is a  $P$ -Galois extension of  $k((s))$ . Pick a non-branch point  $\delta_0 \neq \xi_0$  distinct from  $\delta_1, \dots, \delta_m$ , and let  $W_0 = W$  and  $P_0 = P$ . By [Ha2, Theorem 5.6], there is an irreducible  $\tilde{G}$ -Galois cover  $\tilde{Y}_s \rightarrow X_s := X \times_k k((s))$  that dominates the  $G$ -Galois cover  $Y_s := Y \times_k k((s)) \rightarrow X_s$ , such that  $\tilde{Y}_s \rightarrow Y_s$  is étale away from  $\xi_{0,s} := \xi_0 \times_k k((s))$ , and the fibre over  $\delta_{i,s} := \delta_i \times_k k((s))$  consists of a disjoint union of copies of the  $P_i$ -Galois cover  $\text{Spec } \mathcal{K}_i$  (for  $i = 0, \dots, m$ ). The normalization  $\tilde{\mathcal{Y}}_s$  of  $\mathcal{X}_s := X \times_k k[[s]]$  in  $\tilde{Y}_s$  is an irreducible  $\tilde{G}$ -Galois cover of  $\mathcal{X}_s$ , proper over  $k[[s]]$ . So by [Gr1, Proposition 5.5.1], the closed fibre of  $\tilde{\mathcal{Y}}_s$  is connected.

Let  $\mathcal{X}_t$  be the blow-up of  $X \times_k k[[t]]$  at the points  $\delta_0, \dots, \delta_m$  on the closed fibre  $(t = 0)$ . We may identify the proper transform of  $(t = 0)$  with  $X$ , and regard each exceptional divisor  $S_i$  as a copy of  $S$ , meeting  $X$  at the point  $\delta_i$  (corresponding to the point  $s = 0$  on  $S$ ). So  $\mathcal{X}_t$  is a projective  $k[[t]]$ -curve whose general fibre is  $X_t := X \times_k k((t))$  and whose closed fibre consists of  $X$  and the exceptional divisors  $S_i$ . By formal patching [HS, Cor. to Theorem 1], there is a  $\Gamma$ -Galois cover  $\mathcal{Z}_t \rightarrow \mathcal{X}_t$  whose formal completion along  $X$  is a disjoint union of copies of the  $\tilde{G}$ -Galois cover  $\tilde{\mathcal{Y}}_s \rightarrow \mathcal{X}_s$ , indexed by the cosets of  $\tilde{G}$ ; whose fibre over the exceptional divisor  $S_i$  (for  $i = 0, \dots, m$ ) is a union of copies of the  $Q$ -Galois cover  $W_i \rightarrow S$  that are indexed by the cosets of  $Q$  and are disjoint away from  $\delta_i$ ; and such that the identity copies of  $\tilde{Y}_s$  and  $W_i$  meet at a point  $\zeta_i$  over  $\delta_i$  on  $X$ . Let  $\mathcal{Z}_0$  be the inverse image of  $\mathcal{X}_0 := X \cup S_0$  under  $\mathcal{Z}_t \rightarrow \mathcal{X}_t$ . Then the connected component of  $\zeta_0$  in  $\mathcal{Z}_0$  is all of  $\mathcal{Z}_0$ , since  $Q$  and  $\tilde{G}$  generate  $\Gamma$ . That is,  $\mathcal{Z}_0$  is connected.

Recall that  $\mathcal{X}_t$  is the blow up of  $X \times_k k[[t]]$  at  $\delta_0, \delta_1, \dots, \delta_m$  (on the closed fibre). Now consider the blow up  $\hat{X}_t$  of  $X \times_k k[[t]]$  at  $\delta_1, \dots, \delta_m$  (omitting  $\delta_0$ ). In  $\hat{X}_t$ , identify the proper transform of  $(t = 0)$  under  $\hat{X}_t \rightarrow X \times_k k[[t]]$  with  $X \subset \hat{X}_t$ . Then  $\mathcal{X}_t$  is the blow-up of  $\hat{X}_t$  at  $\delta_0$ . Let  $\hat{Z}_t$  be the normalization of  $\hat{X}_t$  in  $\mathcal{Z}_t$ . Thus  $\mathcal{Z}_t \rightarrow \mathcal{X}_t$  is the normalized pullback of  $\hat{Z}_t \rightarrow \hat{X}_t$ . So for  $i = 1, \dots, m$ , the fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  over  $S_i$  is the same as that of  $\mathcal{Z}_t \rightarrow \mathcal{X}_t$  over  $S_i$ , viz. a union of copies of  $W_i \rightarrow S$  as asserted in Proposition 2.2.

Under the blow-up morphism  $\mathcal{X}_t \rightarrow \hat{X}_t$ , the closed set  $\mathcal{X}_0 \subset \mathcal{X}_t$  is the inverse image of  $X \subset \hat{X}_t$ . Also,  $\mathcal{Z}_0$  is the inverse image under  $\mathcal{Z}_t \rightarrow \hat{Z}_t$  of the fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  over  $X \subset \hat{X}_t$ . But  $\mathcal{Z}_0$  is connected. Hence so is the fibre over  $X \subset \hat{X}_t$ , the proper transform of  $(t = 0)$ . Thus the closed fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  is connected, and hence so is the general fibre  $Z_t \rightarrow X_t = X \times_k k((t))$ . The  $G$ -Galois cover  $\mathcal{Z}_t/Q$  of  $\mathcal{X}_t$  is just  $\mathcal{Y}_t := Y \times_X \mathcal{X}_t$ , since this is true along the formal completion along  $X$  and along the  $S_i$ 's; and  $\mathcal{Z}_t \rightarrow \mathcal{Y}_t$  is étale away from  $\xi_0 \times_k k[[t]]$  and the closed fibre. So  $Z_t \rightarrow X_t$  factors through  $Y_t$ , and  $Z_t \rightarrow Y_t$  is étale away from  $\xi_{0,t} := \xi \times_k k((t))$ . So it is a  $\Gamma$ -Galois cover. Moreover  $Z_t$  is normal since  $\hat{Z}_t$  is. So  $Z_t$  is irreducible, and  $\hat{Z}_t$  is the normalization of  $\hat{X}_t$  in  $Z_t$  (and so agrees with the definition of  $\hat{Z}_t$  in the statement of Proposition 2.2). Finally, if  $K$  is a non-trivial finite extension of  $k((t))$ , then the normalization of  $\hat{X}_t$  in  $X_K := X \times_k K$  has the property that its fibre over the generic point of  $S_1$  is totally ramified; whereas the fibre of  $\hat{Z}_t$  there is totally split. So  $Z_t$  is linearly disjoint from  $X_K = X_t \times_{k((t))} K$  over  $X_t$ , and thus the  $k((t))$ -curve  $Z_t$  is absolutely irreducible.  $\square$

**Corollary 2.3.** *In Proposition 2.1, let  $\delta \in X$  be a non-branch point of  $Y \rightarrow X$ , and let  $\hat{X}_t$  be the blow up of  $X \times_k k[[t]]$  at the point  $\delta$  on the closed fibre. Then we may choose  $Z_t$  such that the normalization  $\hat{Z}_t$  of  $\hat{X}_t$  in  $Z_t$  is unramified over the exceptional divisor away from where it meets the proper transform of  $(t = 0)$ , and such that the fibre of  $\hat{Z}_t$  over this proper transform is connected.*

*Proof.* In Proposition 2.2, take  $m = 1$  and  $\delta_1 = \delta$ . Take  $W_1 \rightarrow S$  to be an arbitrary  $Q$ -Galois cover (e.g. the trivial cover, composed of a disjoint union of copies of  $S$ ). Then Proposition 2.2 gives us a choice of  $Z_t \rightarrow X_t$  satisfying the conclusion there. In particular, the fibre of  $\hat{Z}_t \rightarrow \hat{X}_t$  over the exceptional divisor is a union of copies of  $W_1 \rightarrow S$  away from the point  $\delta$ . So this fibre is étale there.  $\square$

**Remark.** In the above proof of Propositions 2.1 and 2.2, the point  $\delta_0$  is what holds the  $\Gamma$ -Galois cover together, since a  $\tilde{G}$ -Galois cover and a  $Q$ -Galois cover meet at a point  $\zeta_0$  over it. On the other hand, the locus  $\{\delta_1, \dots, \delta_m\}$  (or just  $\delta$  in the Corollary) is where the constructed cover from Proposition 2.1 is blown up (for use in the lemma below).

### Section 3. Proof of the Main Theorem.

In this section we prove the main theorem of the paper, that Abhyankar's local conjecture holds in dimension  $> 1$  if only one coordinate hyperplane is deleted. The proof relies on the previous section together with a formal patching construction applied to a blow-up of a local scheme. It suffices to work in dimension 2, and afterwards to pass to dimension  $n$ .

Specifically, using Corollary 2.3, we prove

**Theorem 3.1.** *Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $\Gamma$  be a finite group, let  $Q = p(\Gamma)$ , and suppose that  $C := \Gamma/Q$  is cyclic. Then there is a Galois étale cover of  $\text{Spec } k[[x, y]][1/y]$  with Galois group  $\Gamma$ .*

The key step in the proof of Theorem 3.1 is the following

**Lemma 3.2.** *Under the hypotheses of Theorem 3.1, suppose that  $\Gamma = Q \rtimes C$ , and that  $C \subset \Gamma$  normalizes a Sylow  $p$ -subgroup of  $Q$ . Then there is a normal connected  $\Gamma$ -Galois cover  $Z \rightarrow \text{Spec } k[[x, y]]$  which is étale away from  $(xy = 0)$ ; such that  $C$  is an inertia group over the generic point of  $(x = 0)$ ; and such that the  $C$ -Galois cover  $Z/Q \rightarrow \text{Spec } k[[x, y]]$  is totally ramified over the generic point of  $(y = 0)$ .*

*Proof of Lemma 3.2.* Let  $R = k[[z]]$ , and let  $P_t = \mathbb{P}_R^1$ , the projective  $t$ -line over  $R$ . Thus we may regard  $\mathbb{P}_K^1$  as the generic fibre of  $P_t$ , where  $K = k((z))$ . Let  $\tilde{P}$  be the blow-up of  $P_t$  at the point  $\xi \in P_t$  where  $(t = z = 0)$ ; let  $E \subset \tilde{P}$  be the exceptional divisor of the blow-up; and let  $T \subset \tilde{P}$  be the proper transform of the closed fibre of  $P_t$ . Also, let  $\tilde{\xi} \in \tilde{P}$  be the point where  $E$  meets  $T$  (and which lies over  $\xi \in P_t$ ). Consider the parameter  $x = z/t$  on  $E$ . Thus  $x = 0$  at the point  $\tilde{\xi}$ ; and we may identify  $E$  with the projective  $x$ -line  $\mathbb{P}_k^1$  over  $k$ , or equivalently with the closed fibre of the projective  $x$ -line  $P_x$  over  $R$ . Note here that the blow-up  $\tilde{P}$  is a closed subset of  $P_t \times_k \mathbb{P}_k^1 = P_t \times_R P_x$ . Here the second projection  $\pi : \tilde{P} \rightarrow P_x$  is a birational isomorphism which is an isomorphism away from  $T \subset \tilde{P}$  (and blows down  $T$ ).

Let  $n$  be the order of  $C$ . Let  $W_0 \rightarrow \mathbb{P}_k^1$  be the  $C$ -Galois cover of the projective  $t$ -line that is given generically by  $w^n = t - 1$ . This cover is branched precisely over  $t = 1, \infty$ , where it is totally ramified. Since  $C$  normalizes a Sylow  $p$ -subgroup of the quasi- $p$  group  $Q$ , we may apply Proposition 1 above to this  $C$ -Galois cover, to the point  $\xi_0 : (t = 1)$  on  $\mathbb{P}_k^1$ , and to the group  $\Gamma$ , using the Laurent series field  $K = k((z))$ . The conclusion is that there is a normal absolutely irreducible  $Q$ -Galois cover  $U^\circ \rightarrow W_0 \times_k K$  that is étale away from the fibre over  $(t = 1)$ , and such that  $U^\circ \rightarrow P^\circ := \mathbb{P}_K^1$  is  $\Gamma$ -Galois. In particular,  $C$  is an inertia group over  $t = \infty$ , and the  $C$ -Galois cover  $U^\circ/Q \approx W_0 \times_k K$  of  $P^\circ$  is totally ramified over the point  $(t = 1)$ . Let  $U$  be the normalization of  $P_t$  in  $U^\circ$ . So  $U \rightarrow P_t$  is a connected normal  $\Gamma$ -Galois cover which is étale away from  $(z = 0)$ ,  $(t = 1)$ , and  $(t = \infty)$ , and whose inertia groups over the generic point of  $(t = \infty)$  are  $n$ -cyclic.

Moreover, using the Corollary above with  $\delta$  taken as the point ( $t = 0$ ) on  $\mathbb{P}_k^1$ , we may choose  $U^\circ$  above with two additional properties. Namely, let  $\tilde{U} \rightarrow \tilde{P}$  be the normalized pullback of  $U \rightarrow P_t$  with respect to  $\tilde{P} \rightarrow P_t$ , and let  $\tilde{U}_0$  be the fibre of  $\tilde{U}$  over  $T$ . Then we may choose  $U^\circ$  such that  $\tilde{U}_0$  is connected and such that  $\tilde{U}$  is étale over  $E$  away from  $\tilde{\xi}$ . So  $\tilde{U} \rightarrow \tilde{P}$  is ramified only over  $t = 1, \infty$  and over  $T$ .

We next consider a  $\Gamma$ -Galois cover of  $P_x$  whose local behavior at ( $x = z = 0$ ) will enable us to obtain the local cover asserted in the statement of the lemma. Namely, let  $V$  be the normalization of  $P_x$  in  $\tilde{U}$ , relative to the morphism  $\tilde{U} \rightarrow \tilde{P} \rightarrow P_x$ . This space is the same as the normalization of  $P_x$  in the general fibre of  $\tilde{U}$ , which is an irreducible normal  $\Gamma$ -Galois cover of the generic fibre of  $P_x$ . Thus  $V$  is an irreducible normal  $\Gamma$ -Galois cover of  $P_x$ . This cover is branched only over  $t = 1, \infty$ , since  $\tilde{U} \rightarrow \tilde{P}$  is ramified only there and over  $T$ , and since  $T$  maps to the closed point ( $x = z = 0$ ) under  $\tilde{P} \rightarrow P_x$ . Here the locus of  $t = 1$  on  $P_x$  is the locus of  $x = z$ , and the locus of  $t = \infty$  on  $P_x$  is the locus of  $x = 0$ . (Note that both of these loci contain the closed point ( $x = z = 0$ ).) So the general fibre of  $V \rightarrow P_x$  is an irreducible  $\Gamma$ -Galois cover of the proj! ictive  $x$ -line over  $K$ , branched only over  $x = 0, z$ . Moreover this cover is tamely ramified over the generic point of  $x = 0$  with  $C$  as an inertia group, and the  $C$ -Galois subcover  $V/Q$  is totally ramified over the generic point  $x = z$ , by the corresponding facts for  $U$  and hence for  $\tilde{U}$  over  $t = 1, \infty$ .

Now under the surjection  $\tilde{U} \rightarrow V$ , the connected closed set  $\tilde{U}_0$  is the inverse image of the fibre of  $V$  over the point ( $x = z = 0$ ). So that fibre is connected; i.e. the irreducible normal  $\Gamma$ -Galois cover  $V \rightarrow P_x$  is totally ramified over ( $x = z = 0$ ). So the pullback of  $V$  under the morphism  $\text{Spec } k[[x, z]] \rightarrow P_x$  is also an irreducible normal  $\Gamma$ -Galois cover, branched only over  $x = 0$  and  $x = z$ , with  $C$  as an inertia group over  $x = 0$  and with the quotient modulo  $Q$  being a  $C$ -Galois cover with total ramification over  $x = z$ . Setting  $y = z - x$  completes the proof of the lemma.  $\square$

Using Lemma 3.2, we complete the proof of Theorem 3.1:

*Proof of Theorem 3.1.* Let  $n$  be the order of  $C$ . Thus  $n$  is prime to  $p$ . By [Ha1, Lemma 5.3] (or by using the Schur-Zassenhaus Theorem in group theory [Go]), there is a prime-to- $p$  cyclic supplement  $C' \subset \Gamma$  to  $Q$  in  $\Gamma$  such that  $C'$  normalizes a Sylow  $p$ -subgroup of  $Q$ . Thus  $\Gamma$  is a quotient of the semidirect product  $\Gamma' := Q \rtimes C'$ , and  $Q = p(\Gamma')$ . So it suffices to prove the result for  $\Gamma'$ . Replacing  $\Gamma, C$  by  $\Gamma', C'$ , we may assume that  $\Gamma = Q \rtimes C$ , and view  $C$  as a subgroup of  $\Gamma$  that normalizes a Sylow  $p$ -subgroup of  $Q$ .

So the lemma applies. Thus there is a normal connected  $\Gamma$ -Galois cover  $Z \rightarrow S := \text{Spec } k[[s, y]]$  which is étale away from ( $sy = 0$ ), such that  $C$  is an inertia group over the generic point of ( $s = 0$ ), and such that  $W := Z/Q \rightarrow S$  is totally ramified over ( $y = 0$ ). Here  $W \rightarrow S$  is a normal connected  $C$ -Galois cover of  $S$ , branched only over ( $sy = 0$ ); and  $Z \rightarrow W$  is a normal connected  $Q$ -Galois cover.

Let  $X = \text{Spec } k[[x, y]]$ , and consider the morphism  $X \rightarrow S$  given by  $x^n = s$ . Let  $Z' \rightarrow X$  [resp.  $W' \rightarrow X$ ] be the normalized pullback of  $Z \rightarrow S$  [resp. of  $W \rightarrow S$ ] with respect to this morphism. Thus  $W' = Z'/Q$ , and  $Z'$  is the normalized fibre product of  $W'$  and  $Z$  over  $W$ . Since  $X \rightarrow S$  is étale over the generic point of  $(y = 0)$ , whereas  $W \rightarrow S$  is totally ramified there, it follows that  $X$  and  $W$  are linearly disjoint over  $S$ , and hence  $W'$  is irreducible.

By Abhyankar's Lemma, the cover  $Z' \rightarrow X$  is étale over the generic point of  $(x = 0)$ , and so this cover is branched only over  $(y = 0)$ . Now  $W' \rightarrow W$  is  $C$ -Galois, where  $C$  is of order prime-to- $p$ . But  $Z \rightarrow W$  is  $Q$ -Galois, where  $Q$  is a quasi- $p$  group. So  $C$  and  $Q$  have no non-trivial common quotients, and the two covers  $Z \rightarrow W$  and  $W' \rightarrow W$  dominate no non-trivial common subcover of  $W$ . Since these two covers are also Galois, it follows that they are linearly disjoint. Hence their normalized fibre product  $Z'$  is irreducible. The restriction of the  $\Gamma$ -Galois cover  $Z' \rightarrow X$  to  $y \neq 0$  is then the asserted cover of  $\text{Spec } k[[x, y]][1/y]$ .  $\square$

Theorem 3.1 is the case of dimension 2 in the Main Theorem. The general case is now immediate:

**Main Theorem 3.3.** *Abhyankar's Local Conjecture on Galois groups over  $\hat{R}_{n,r} = k[[x_1, \dots, x_n]][(x_1 \cdots x_r)^{-1}]$  (with  $n > 1$  and  $1 \leq r \leq n$ ) holds if and only if  $r = 1$ .*

*Proof.* As noted in the introduction, the case of  $r > 1$  was disproven in [HP]. So it remains to prove the case  $r = 1$ ; i.e. that a finite group  $G$  is the Galois group of an unramified extension of  $\hat{R}_{n,1} = k[[x_1, \dots, x_n]][x_1^{-1}]$  if and only if  $G/p(G)$  is cyclic. Theorem 3.1 is the case  $n = 2$ . For the general case, let  $G$  be a finite group with  $G/p(G)$  cyclic, and let  $S$  be a Galois étale extension of  $R_{n,1}$  with group  $G$ , given by Theorem 3.1. Then  $R_{n,1}$  is algebraically closed in  $R_{n,r}$ , and  $R_{n,r}$  is separable over  $R_{n,1}$ ; so  $R_{n,r}$  is linearly disjoint from  $S$  over  $R_{n,1}$  [FJ, Lemma 9.7]. Thus  $R_{n,r} \otimes_{R_{n,1}} S$  is a domain, and is an unramified Galois extension of  $R_{n,r}$  with group  $G$ .  $\square$

Thus we conclude that the Galois groups over  $k[[x_1, \dots, x_n]][x_1^{-1}]$  are precisely the cyclic-by-quasi- $p$  groups:

**Corollary 3.4.** *If  $1 \rightarrow Q \rightarrow G \rightarrow C \rightarrow 1$  is an exact sequence of finite groups, with  $Q$  a quasi- $p$  group and  $C$  a cyclic group, then  $G$  is the Galois group of an unramified extension of  $\hat{R}_{n,1} = k[[x_1, \dots, x_n]][x_1^{-1}]$  for all  $n > 1$ . Conversely, every Galois group  $G$  over  $\hat{R}_{n,1}$  is of this form.*

*Proof.* Since  $Q$  is a quasi- $p$  group, it is contained in  $p(G)$ . Hence  $G/p(G)$  is a quotient of  $C$ , and thus is cyclic. So  $G$  is a Galois group over  $k[[x_1, \dots, x_n]][x_1^{-1}]$  by the Main Theorem. This proves the forward direction. The converse is also immediate from the Main Theorem, by taking  $Q = p(G)$  and  $C = G/p(G)$ .  $\square$

## Section 4. Another viewpoint.

In this section, we sketch an alternative construction to prove Theorem 3.1 and hence the Main Theorem. The presentation here, which relies on a series of blowings-up, may be viewed as geometrically more intuitive than the earlier argument; but filling in all the details here (particularly concerning the use of the patching result of [HS]) would produce a longer argument than the one in the previous section.

The basic idea of the proof here is to start with a cyclic cover of the base space  $\text{Spec } k[[x, y]]$  branched only along  $(x = 0)$ ; to blow up the base, obtaining an exceptional divisor  $E$ ; and then to use Abhyankar's Conjecture for the affine line [Ra] together with formal patching [HS] on  $E$  and the existence of solutions to  $p$ -embedding problems [H3] in order to enlarge the Galois group of the cover by a quasi- $p$  group. The difficulty is that doing this would cause the resulting cover of  $\text{Spec } k[[x, y]]$  (after blowing back down) to be ramified along both axes; so instead we blow up a series of times, to avoid obtaining extra ramification. (The corresponding trick in the first proof of Theorem 3.1, given in Section 3, was to consider the morphism  $X \rightarrow S$  given by  $x^n = s$ .)

We begin with a definition and a lemma. As before, let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ . In  $k$ , fix a compatible system  $\{\zeta_n\}_{(p,n)=1}$  of roots of unity in  $k$  (i.e. such that  $\zeta_{mn}^m = \zeta_n$ ).

**Definition 4.1** (a) Let  $V \rightarrow X$  be a  $G$ -Galois cover of normal  $k$ -schemes. Let  $\alpha$  be a point of  $X$  of codimension 1 at which  $X$  is regular and over which the cover is tamely ramified, say of ramification index  $m$ . Let  $x \in \hat{\mathcal{O}}_{X,\alpha}$  be a uniformizer at  $\alpha$ . If  $\beta$  is a point of  $V$  over  $\alpha$ , and  $v \in \hat{\mathcal{O}}_{V,\beta}$  is a uniformizer at  $\beta$  such that  $v^m = x$ , then the *canonical generator of inertia of  $V \rightarrow X$  at  $\beta$*  is the unique element  $g \in G$  of the inertia group at  $\beta$  such that  $g(v) = \zeta_m v$ . An element  $g \in G$  is a *canonical generator of inertia of  $V \rightarrow X$  over  $\alpha$*  if it is a canonical generator of inertia at some point  $\beta$  over  $\alpha$ .

(b) Let  $X$  be a regular  $k$ -scheme, let  $V$  be a normal  $k$ -scheme, and let  $V \rightarrow X$  be a  $G$ -Galois cover. Let  $\xi \in X$  be a closed point in the branch locus  $B$ , at which  $B$  has at most normal crossings. Let  $\nu$  be a point of  $V$  over  $\xi$ , and let  $D_1, \dots, D_r$  be the irreducible components of the branch locus of  $\hat{V} = \text{Spec } \hat{\mathcal{O}}_{V,\nu} \rightarrow \hat{X} = \text{Spec } \hat{\mathcal{O}}_{X,\xi}$ . Let  $g_i \in G$  be the canonical generator of inertia of  $\hat{V} \rightarrow \hat{X}$  at the generic point of  $D_i$ . Then  $g_1, \dots, g_r$  are the *canonical generators of inertia of  $V \rightarrow X$  at  $\nu$* .  $\square$

Note that in (a) of the definition, once one chooses  $x$ , a uniformizer  $v$  as above always exists. Moreover the canonical generator of inertia at  $\beta$  depends only on  $\beta$ , not on the choice of  $x$  or  $v$ . The canonical generators of inertia over  $\alpha$  form a conjugacy class in  $G$ . So in (b), the  $g_i$ 's are each determined up to conjugation in the inertia group at  $\nu$  (which is the Galois group of  $\hat{\mathcal{O}}_{V,\nu}$  over  $\hat{\mathcal{O}}_{X,\xi}$ ). But that group is abelian, of order prime to  $p$ , and of rank at most  $r$ . (This standard fact follows from Abhyankar's Lemma; e.g. see [HP,

Proposition 3.1].) So actually the  $g_i$ 's in (b) are uniquely determined by  $\nu$ .

**Lemma 4.2** *Suppose that  $V \rightarrow X$  is a  $G$ -Galois cover of  $k$ -schemes, with  $X$  regular and  $V$  normal. Let  $\xi$  be a closed point of  $X$  at which the branch locus  $B$  has at most normal crossings, and let  $\nu \in V$  be a point lying over  $\xi$ , with canonical generators of inertia  $g_1, \dots, g_r$ . Let  $\hat{X}$  be the blow-up of  $X$  at  $\xi$ , and let  $\hat{V}$  be the normalization of  $V \times_X \hat{X}$ .*

(a) *Then  $\hat{V} \rightarrow \hat{X}$  is tamely ramified over the exceptional divisor  $E$ .*

(b) *Let  $\epsilon$  be the generic point of  $E \subset \hat{X}$ , and let  $\phi \in \hat{V}$  be a point lying over  $\epsilon$  that maps to  $\nu$  under  $\hat{V} \rightarrow V$ . Then the canonical generator of inertia at  $\phi$  is  $g_1 \cdots g_r$ .*

Note that the element  $\prod g_i$  in (b) of the lemma is independent of the order in which the  $g_i$ 's are taken. This is because the  $g_i$ 's lie in an abelian group (viz. the inertia group at  $\nu$ ), by the comment preceding the statement of the lemma.

*Proof of Lemma 4.2.* Since the result is local, we are reduced to the special case that  $X = \text{Spec } k[[x_1, x_2, \dots, x_n]]$ , with  $\xi$  being the closed point  $(x_1, x_2, \dots, x_n)$ , and where  $V \rightarrow X$  is branched only at  $D_1 \cup \dots \cup D_r$  for some  $r \leq n$ , where  $D_i := (x_i = 0)$ . This cover is totally ramified over  $\xi$  since  $X$  is completely local and since  $k$  is algebraically closed; so its Galois group  $G$  is the inertia group over  $\xi$ .

First consider the case that  $V \rightarrow X$  is the cover  $S \rightarrow X$  given by  $s_i^m = x_i$  ( $1 \leq i \leq r$ ). This cover is Galois with group  $A = (C_m)^r = \langle a_1, a_2, \dots, a_r \rangle$ , where the action is given by  $a_i(s_i) = \zeta_m s_i$  and  $a_i(s_j) = s_j$  for  $i \neq j$ . Thus  $a_i$  is the canonical generator of inertia of  $S \rightarrow X$  over  $D_i$ . Let  $\hat{X}$  be the blow-up of  $X$  at  $\tau$ , with exceptional divisor  $E \approx \mathbb{P}^{n-1}$ . The function field of  $E$  is generated by  $z_j = x_j/x_1$ , for  $j = 2, \dots, n$ . Let  $\hat{S}$  be the normalized pullback of  $S \rightarrow X$  via  $\hat{X} \rightarrow X$ ; thus  $\hat{S} \rightarrow \hat{X}$  is an  $A$ -Galois cover. Since  $A$  is of order prime to  $p$ , all ramification is tame. Let  $F$  be a component of the fibre of  $\hat{S} \rightarrow \hat{X}$  over  $E$ . Thus the function field of  $F$  is generated by  $w_j = s_j/s_1$ , for  $j = 2, \dots, n$ . Here  $w_j^m = z_j$ . Now  $a_1^{i_1} a_2^{i_2} \dots a_r^{i_r}(w_j) = \zeta_m^{i_j - i_1} w_j$ . Thus the parameter  $w_j$  is fixed exactly when  $i_j = i_1$ , so the inertia group at the generic point  $\phi$  of  $F$  is the cyclic group  $\langle a_1 a_2 \dots a_r \rangle$ . The element  $s_1$  is a local uniformizer at the generic point  $\epsilon$  of  $E$  in  $\hat{X}$ ; and for each  $d$  in the inertia group  $\langle a_1 a_2 \dots a_m \rangle$ , we have  $d = a_1^{i_1} a_2^{i_2} \dots a_r^{i_r}$  and  $d(s_1) = a_1^{i_1} a_2^{i_2} \dots a_r^{i_r}(s_1) = \zeta_m^{i_1} s_1$ . Thus the canonical generator of inertia is  $d = a_1 a_2 \dots a_r$ . This proves the lemma for the cover  $S \rightarrow X$ .

Now consider the case of a more general cover  $V \rightarrow X = \text{Spec } k[[x_1, x_2, \dots, x_n]]$ . By [HP, Proposition 3.1],  $G$  is abelian of rank  $r \leq n$ , and  $V \rightarrow X$  is dominated by the  $A$ -Galois cover  $S \rightarrow X$  of the previous paragraph (for some  $r \leq n$ ). The canonical generator of inertia of  $V \rightarrow X$  over the generic point of  $D_i$  [resp. of  $E$ ] is the image, under the quotient map  $A \rightarrow G$ , of the canonical generator of inertia of  $S \rightarrow X$  there — i.e. the image of  $a_i$  [resp. of  $a_1 a_2 \dots a_r$ ] in  $G$ . So the canonical generator of inertia of  $V \rightarrow X$  over  $\epsilon$  is as claimed.  $\square$

We now use this lemma to provide the alternative construction to prove Theorem 3.1 and hence the Main Theorem.

*Alternative proof sketch of Theorem 3.1.* As in the previous proof of this theorem, we immediately reduce to the case that  $\Gamma = Q \rtimes C$ , where  $Q$  is a quasi- $p$  group,  $C$  is a cyclic group of order  $n$  prime to  $p$ , and  $C \subset \Gamma$  normalizes a Sylow  $p$ -subgroup  $P$  of  $\Gamma$  (or equivalently, of  $Q$ ).

Let  $X$  be the projective  $x$ -line over  $k$ , and  $\mathcal{X} = X \times_k k[[y]]$ . Identify  $X$  with the closed fiber of  $\mathcal{X}$ . Consider the  $C$ -Galois cover  $\mathcal{W} \rightarrow \mathcal{X}$  defined by  $w^n = x$ . This is ramified only at the point  $\tau$  where  $x = 0$  and at the point  $\sigma$  where  $x = \infty$ ; and over both points the cover is totally ramified. Let  $c$  be the canonical generator of inertia over  $x = 0$ . So  $C = \langle c \rangle$ . Let  $Y = \tau \times_k k[[y]] \subset \mathcal{X}$  and  $S = \sigma \times_k k[[y]] \subset \mathcal{X}$ . Let  $\mathcal{X}_0 = \mathcal{X}$  and for each  $i = 1, 2, \dots, n$ , let  $\mathcal{X}_i$  be the blow-up of  $\mathcal{X}_{i-1}$  at  $\tau$  with exceptional divisor  $E_i$ . Here we inductively identify  $X, E_1, \dots, E_{i-1}, Y, S \subset \mathcal{X}_{i-1}$  with their proper transforms in  $\mathcal{X}_i$ , and identify  $\tau$  with the point of  $\mathcal{X}_i$  where  $Y$  meets  $E_i$ . Let  $\mathcal{W}_i \rightarrow \mathcal{X}_i$  be the normalized pullback of  $\mathcal{W}_{i-1}$  with respect to  $\mathcal{X}_i \rightarrow \mathcal{X}_{i-1}$ . Then this is a  $C$ -Galois cover ramified only over  $E_1, E_2, \dots, E_i, Y, S$ . The canonical generator of inertia over  $Y$  is  $c$ ; and by the lemma (and induction on  $i$ ), the canonical generators of inertia over  $E_i$  is  $c^i$ . Notice that the canonical generator of inertia of  $E_n$  is  $c^n = 1$ , so  $\mathcal{W}_n \rightarrow \mathcal{X}_n$  is unramified over the generic point of  $E_n$ .

Let  $\delta$  be the point where  $E_n$  meets  $E_{n-1}$ , and let  $\xi$  be a point of  $E_n - \{\tau, \delta\}$ . Let  $\mathcal{X}_{n+1}$  be the blow-up of  $\mathcal{X}_n$  at  $\xi$ , with exceptional divisor  $E_{n+1}$ . Identify  $\xi$  with the point of  $\mathcal{X}_{n+1}$  where  $E_{n+1}$  meets  $E_n$  and identify  $X, Y, S, E_i$  with their proper transforms in  $\mathcal{X}_{n+1}$ . By [Ra] there exists a  $Q$ -Galois cover  $Z \rightarrow E_{n+1}$  branched only at the point  $\xi$  such that the inertia groups over  $\xi$  are conjugate to  $P$ .

Let  $\mathcal{E}$  be the formal completion of  $\mathcal{X}_{n+1}$  along the closed subset  $E_1 \cup E_2 \cup \dots \cup E_n$ , and let  $\mathcal{W}_{\mathcal{E}} \rightarrow \mathcal{E}$  be the pullback of  $\mathcal{W}_{n+1} \rightarrow \mathcal{X}_{n+1}$  via  $\mathcal{E} \rightarrow \mathcal{X}_{n+1}$ . By applying [Ha2, Theorem 5.6] to the open fibre of the  $C$ -Galois cover  $\mathcal{W}_{\mathcal{E}} \rightarrow \mathcal{E}$ , we obtain a  $G = P \rtimes C$ -Galois cover  $\mathcal{U} \rightarrow \mathcal{E}$  that dominates  $\mathcal{W}_{\mathcal{E}}$  and is ramified only at  $E_1, E_2, \dots, E_n, Y$ , and such that there is agreement locally between the generic fibre of  $\mathcal{U} \rightarrow \mathcal{E}$  and the  $Q$ -Galois cover  $Z \rightarrow E_{n+1}$  near the point  $\xi$  (i.e. over the generic point of  $\text{Spec } \hat{\mathcal{O}}_{E_{n+1}, \xi}$ , which is a closed point of the generic fibre of  $\mathcal{E}$ ). Moreover, according to [Ha2, Theorem 5.6], we can require that the  $G$ -Galois cover  $\mathcal{U} \rightarrow \mathcal{E}$  also agrees locally with the original  $C$ -Galois cover  $\mathcal{W} \rightarrow X$  near the point  $\chi$  at which  $E_1$  meets  $X$ . Using these local agreements, we may apply formal patching [HS] to obtain a  $\Gamma$ -Galois cover  $\mathcal{V}_{n+1} \rightarrow \mathcal{X}_{n+1}$  whose restrictions agree with the covers  $Z \rightarrow E_{n+1}, \mathcal{U} \rightarrow \mathcal{E}$ , and  $\mathcal{W} \rightarrow X$ , where the first two meet over  $\xi$  and the latter two meet over  $\chi$ . In particular, this cover is unramified over the generic point of  $X$ . Let  $\mathcal{V}$  be the normalization of  $\mathcal{X}$  in  $\mathcal{V}_{n+1}$  (relative to the blow-up morphism  $\mathcal{X}_{n+1} \rightarrow \mathcal{X}$ ). The natural morphism  $\mathcal{V}_{n+1} \rightarrow \mathcal{V}$  is a birational isomorphism, and  $\mathcal{V} \rightarrow \mathcal{X}$  is a  $\Gamma$ -Galois cover ramified only along  $Y$  and totally ramified at  $\tau$ . Let  $\mathcal{X}_{\tau} = \text{Spec } k[[x, y]]$  be the

spectrum of the complete local ring of  $\mathcal{X}$  at  $\tau$ , and let  $\mathcal{V}_\tau$  be the pullback of  $\mathcal{V}$  to  $\mathcal{X}_\tau$ . Thus  $\mathcal{V}_\tau \rightarrow \mathcal{X}_\tau = \text{Spec } k[[x, y]]$  is a  $\Gamma$ -Galois cover that is branched totally at the closed point  $\tau$ , and is unramified away from  $(x = 0)$ . This completes the construction, yielding Theorem 3.1 and hence the Main Theorem in general.  $\square$

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