

Course Notes for Math 320:
Fundamentals of Mathematics
Chapter 4: Relations, Equivalence Relations and
Congruence.

March 14, 2006

1 Relations

Definition 1.1. (*Kazimierz Kuratowski*) Let S be a set and let a and b be members of S . The **ordered pair** (a, b) is the set $\{\{a\}, \{a, b\}\}$. The element a is called the **first term** of (a, b) and the element b is called the **second term** of (a, b) .

Lemma 1.2. Let (a, b) and (c, d) be ordered pairs. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Definition 1.3. 1. A **relation** is a collection of ordered pairs.

2. If R is a relation then the collection of all first terms is called the **domain** and the collection of all second terms is called the **range**.

3. If x is in the domain of R let $R[x] = \{y \in \text{range of } R \mid (x, y) \in R\}$.

4. If A and B are sets then $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. This is a relation, called the **Cartesian product**.

Example 1.4. Find the domain A and range B and if possible express as $A \times B$.

1. $R_1 = \{(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), (Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), (Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}$.

What is $R_1[\text{Giuseppe}]$? What is $R_1[\text{Pietro}]$?

2. $R_2 = \{(1, \pi), (3, 1), (5, 1), (2, 2), (\pi, 1)\}$. What is $R_2[1]$?

3. $\mathbf{R}^2 = \{(x, y) | x, y \in \mathbf{R}\}$. What is $\mathbf{R}^2[1]$?

4. $R_3 = \{(x, y) | x, y \in R \text{ and } y = 2x\}$. What is $R_3[1]$?

5. $R_4 = \{(x, y) | x, y \in R \text{ and } y = x^2\}$. What is $R_4[1]$?

6. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Find $A \times B$.

7. Let $A = \{x | 1 < x \leq 2\}$, $B = \{1, 2, 3\}$, and $C = \{x | 2 < x < 3\}$. Compute $A \times B$, $B \times A$, $A \times C$, and $A \times \mathbf{R}$.

8. Let $A = \{1\}$, $B = \{3\}$, $C = \{4\}$ and $D = \{2\}$. Compute $(A \times B) \cup (C \times D)$ and $(A \cup C) \times (B \cup D)$.

Proposition 1.5. *If A , B , C , and D are sets then the following hold:*

1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3. $(A - B) \times B = (A \times B) - (B \times B)$ (F&P Prop 4.2)
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
5. $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.

space

2 Graphs... ways of depicting relations

Graphs are a natural way of describing certain relations and motivating proofs.

1.
$$R_1 = \{(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), (Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), (Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}.$$

2.
$$R_2 = \{(1, 2), (3, 1), (5, 1), (2, 2), (2, 1)\}.$$

3.
$$R_2 \cup R_2^{-1}.$$

4.
$$\mathbf{R}^2 = \{(x, y) | x, y \in \mathbf{R}\}.$$

5.
$$R_3 = \{(x, y) | x, y \in R \text{ and } y = 2x\}.$$

6.
$$R_4 = \{(x, y) | x, y \in R \text{ and } y = x^2\}.$$

CAUTION: Graphs \neq Proofs. Having said that they can be informative.

Definition 2.1. Let R be a relation. Consider the set:

$$R^{-1} = \{(x, y) | (y, x) \in R\}.$$

This set is called the **R -inverse**. We call a relation **symmetric** if $R = R^{-1}$.

Example 2.2. Find R^{-1} . Which are symmetric?

1. $R_1 = \{(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), (Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), (Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}.$

2. $R_2 = \{(1, 2), (3, 1), (5, 1), (2, 2), (2, 1)\}.$

3. $R_2 \cup R_2^{-1}.$

4. $\mathbf{R}^2 = \{(x, y) | x, y \in \mathbf{R}\}.$

5. $R_3 = \{(x, y) | x, y \in R \text{ and } y = 2x\}.$

6. $R_4 = \{(x, y) | x, y \in R \text{ and } y = x^2\}$.

7. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$ Find $A \times B$.

8. Let $A = \{x | 1 < x \leq 2\}$, $B = \{1, 2, 3\}$, and $C = \{x | 2 < x < 3\}$. Compute $A \times B$.

Definition 2.3. Let R be a relation with domain A and range B .

1. The cartesian graph of R is the plot of points $\{(x, y) \in A \times B | (x, y) \in R\}$ on a set of axes with horizontal axis representing A and vertical axis representing B .
2. Suppose that $A, B \in S$. The directed graph of R is a collection points, randomly placed, each representing exactly one element of S and each element of S being represented. Then given points representing $x, y \in S$ we connect them with an arrow from x to y if $(x, y) \in R$.

Remark 2.4. If a relation R is symmetric what can you say about its cartesian or directed graphs?

Proposition 2.5. *Let R and S be relations. Then each of the following statements is true:*

1. $R = (R^{-1})^{-1}$
2. $R \subset S$ if and only if $R^{-1} \subset S^{-1}$
3. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
4. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
5. $R \cup R^{-1}$ is symmetric
6. $R \cap R^{-1}$ is symmetric.

3 Equivalence Relations

Example 3.1. *Some more examples:*

1.
$$R_1 = \{(Pietro, Giuseppe), (Pietro, Matteo), (Giuseppe, Pietro), (Giuseppe, Matteo), (Giuseppe, Marco), (Marco, Giuseppe), (Marco, Matteo), (Pietro, Enrico), (Marco, Enrico), (Giuseppe, Enrico), (Giuseppe, Giuseppe)\}.$$

2. *Consider the following relation R on \mathbf{Z} .*

$$R_2 = \{(x, y) | x, y \in \mathbf{Z} \text{ and } x = -y\}.$$

3. *Consider the following relation R on \mathbf{R} .*

$$R_3 = \{(x, y) | x, y \in \mathbf{R} \text{ and } x - y = n \text{ for some } n \in \mathbf{Z}_{\geq 0}\}.$$

4. *Consider the following relation R on \mathbf{Z} .*

$$R_4 = \{(x, y) | x, y \in \mathbf{Z} \text{ and } x - y \text{ is even}\}.$$

Definition 3.2. Let R be a relation on a set S .

1. We call R **reflexive** on S if for every $x \in S$ we have $(x, x) \in R$.
2. We call R **symmetric** if whenever $(x, y) \in R$ then $(y, x) \in R$.
3. We call R **transitive** if whenever $(x, y), (y, z) \in R$ then $(x, z) \in R$.

Remark 3.3. Caution:

1. To show that a relation R is reflexive, we take *ONE* arbitrary element $x \in S$ and show that $(x, x) \in R$. What should the cartesian graph or directed graph of R look like if it is reflexive?
2. To show that a relation R is symmetric, we take *TWO* arbitrary elements $x, y \in S$, such that $(x, y) \in R$ and we use this to show that $(y, x) \in R$. What should the cartesian graph or directed graph of R look like if it is symmetric?
3. To show that a relation R is transitive, we take *THREE* arbitrary elements $x, y, z \in S$ such that $(x, y), (y, z) \in R$ and use this to show that $(x, z) \in R$. What should the cartesian graph or directed graph of R look like if it is transitive?

Example 3.4. Which of the above is reflexive, symmetric, and/or transitive?

Definition 3.5. A relation that is reflexive, symmetric, AND transitive is called an equivalence relation.

Example 3.6. Which of the above is an equivalence relation?

Remark 3.7. Recall that given a relation R on a set S then $[x] = \{y \in S \mid (x, y) \in R\}$. Or in the 'tilde' notation: $[x] = \{y \in S \mid x \sim y\}$. Now we can rephrase the requirements to be an equivalence relation in terms of this representative $[x]$.

1. For each $x \in S$...
2. If $x \in [y]$ then...
3. If $y \in [x]$ and ...

Definition 3.8. Given an equivalence relation R on a set S and an element $x \in S$, the set $[x]$ is called the **equivalence class** of x .

Group Work problems for Relations, graphs, and equivalence relations:

1. Let $R = \{(a, b) \in \mathbf{N} \times \mathbf{N} \mid a|b\}$. List five members of $R[7]$, and list five members of $R[14]$. For which $n \in \mathbf{N}$ is it true that $R[n] = \mathbf{N}$? Is R symmetric? Is it an equivalence relation?
2. Let $R = \{(x, y \in \mathbf{R} \times \mathbf{R} \mid x^2 - y^2 = 0\}$.
 - (a) Prove that R is an equivalence relation on \mathbf{R} .
 - (b) List all the elements of $[3]$.
3. Let $S = \mathbf{N} \times \mathbf{N}$. For any two members (a, b) and (c, d) of S we will define a relation $(a, b) \sim (c, d)$ if $ad = bc$. Prove that \sim is an equivalence relation.
4. Let $A = \{1, 2, 3, 4, 5\}$. Draw the directed graphs associated with each of the following relations on A . Which relations are equivalence relations?
 - (a) $R = \{(a, b) \in A \times A \mid a|b\}$
 - (b) $R = \{(a, b) \in A \times A \mid a \neq b\}$
 - (c) $R = \{(a, b) \in A \times A \mid a + b \in 2\mathbf{Z}\}$
 - (d) $R = \{(a, b) \in A \times A \mid a + b \text{ odd}\}$
5. Let R be a relation such that $R^{-1} \subset R$. Must R be symmetric? Prove your answer.

4 Congruence or clock arithmetic: Example of equivalence relation

Example 4.1. Consider the following relation R on \mathbf{Z} .

$$R_n = \{(x, y) \mid x, y \in \mathbf{Z} \text{ and } x - y = nk \text{ for some } k \in \mathbf{Z}\}.$$

Group Work

- Let p be an odd prime and let $S = \{[1], [2], \dots, [p-1]\}$ be the set of nonzero equivalence classes mod p .
 - Prove that for any integer a such that $[a] \in S$, we have that $[-a] \in S$ and that $[-a] \neq [a]$.
 - Prove that if $[a] \in S$ and $[b] \in S$, then $[a] \otimes [b] \in S$.
 - Prove that if $[a]^2 = [b]^2$ then $[a] = [b]$ or $[a] = -[b]$.
 - Prove that exactly $(p-1)/2$ members of S have square roots (i.e. $[a]$ has a square root provided that there is an integer x such that $[x]^2 = [a]$).

Hint. First show that there are at most $(p-1)/2$ elements of S with square roots by using that $[x]^2 = [-x]^2$. Then consider the following subset of S : $\{[1]^2, [2]^2, \dots, [(p-1)/2]^2\}$. Show that no two of these can be equal using above parts. This shows that there are at least $(p-1)/2$ elements of S with square roots.
- In the previous problem, instead of taking p to be an odd prime number, take $p = 16$. Show by example that all four statements are then false.