Course Notes for Math 320: Fundamentals of Mathematics Chapter 3: Induction.

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1 Proof by Induction

Definition 1.1. *A subset S of the natural numbers is said to be inductive if $\forall n \in S$ we have $n+1 \in S$.*

Example 1.2. *Which sets are inductive?*

1. *Natural Numbers:* $\mathbf{N} = \{1, 2, 3, 4, ...\}$
2. *Even Natural numbers:* $2\mathbf{N} = \{m \in \mathbf{N} | \exists a \in \mathbf{N} \text{ such that } m = 2a\}$
3. *Natural numbers that are perfect squares*
4. *All natural numbers greater than or equal to 2591*
5. $\{n \in \mathbf{N} | 4^n > n^4\}$
6. $\{n \in \mathbf{N} | 2^{n+1} \leq 3^n\}$
7. $\{n \in \mathbf{N} | 3 | (5^n - 2^n)\}$
8. $\{n \in \mathbf{N} | 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}\}$

Principle 1.3. Principle of Mathematical Induction:
If a subset S of $\mathbf{N}$ is inductive and $1 \in S$ then $S = \mathbf{N}$.

We can use this to prove stuff when we feel ourselves saying... "and you continue in this way" or "repeating this process over and over"...

Proposition 1.4. *(Gauss, age 10)*
For all natural numbers n we have that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Proposition 1.5. *For all natural numbers n we have that $3|(5^n - 2^n)$.*

Proposition 1.6. *For all natural numbers n we have that $(x^n - y^n)$ is divisible by $x - y$, where x and y are any two distinct integers.*

Proposition 1.7. *For all natural numbers n we have that $1^2 + 2^3 + 3^3 + ... + n^3 = \frac{n(n+1)(2n+1)}{6}$*

Proposition 1.8. *For all natural numbers n we have that if A is a finite set with $|A| = n$ then $|P(A)| = 2^n$.*

Remark 1.9. *To prove such results we follow a proof by induction:*

1. *Define a set $S = \{n \in \mathbf{N}| \text{ stuff is true } \}$*
2. *Show that $1 \in S$*
3. *Show that it is inductive.*

Conlcusion: $S = \mathbf{N}$.

More proofs

Example 1.10. *(F&P3.9) What is wrong with the following proof that all members of the human race are of the same sex?*
Proof: *Let*

$$S = \{n \in \mathbf{N} | \text{ in any set of } n \text{ people, all members of the set are the same sex}\}.$$

If we have a set consisting of one person, then clearly all the members of the set are of the same sex, so $1 \in S$. *Suppose that* $n \in S$. *Then in any set of n people, all the members of the set are of the same sex. In order to show that* $n+1 \in S$, *we need to show that in any set of* $n+1$ *people, all the members of the set are of the same sex. Let* A *be a set consisting of* $n+1$ *people. Let's call these people* $a_1, a_2, ..., a_n, a_{n+1}$. *If we send one person out of the room, say* a_1, *then we have a set of n people left in the room, so by the assumption that* $n \in S$, *they are all of the same sex. Now let's bring* a_1 *back into the room and set* a_2 *out. Again there is a set of n people left in the room, so by the assumption that* $n \in S$ *they are all of the same sex. Now observe that everyone in the original set of* $n+1$ *people is of the same sex as* a_3, *so the are all of the same sex. Therefore* $n+1 \in S$. *By the Principle of Mathematical Induction,* $S = N$.

2 Variants on Principle of Mathematical Induction

Example 2.1. *Recall the set* $S = \{n \in \mathbf{N} | 4^n > n^4\}$ *which we believed to be inductive after* $n = 5$*. In fact we think that it contains* $\{n \in \mathbf{N} | n \geq 5\}$*. Can we use PMI for this?*

Principle 2.2. Extended Principle of Mathematical Induction
Let $k \in \mathbf{N}$ *and let* S *be a subset of* $\mathbf{N}$ *such that*

a. $k \in S$*, and*

b. if $n \geq k$ *and* $n \in S$ *then* $n + 1 \in S$*.*

Then $\{n \in \mathbf{N} | n \geq k\} \subset S$*.*

Theorem 2.3. *PMI* $\Rightarrow$ *EPMI*

Proposition 2.4. *For all natural numbers* $n \geq 2$ *we have that* $2^{n+1} \leq 3^n$*. (Hint: EPMI)*

Principle 2.5. Second Principle of Mathematical Induction
Let S *be a subset of* $\mathbf{N}$ *such that*

a. $1 \in S$*, and*

b. For each $n \geq k$ *if* $\{1, 2, 3, ..., n\} \subset S$ *then* $n + 1 \in S$*.*

Then $S = \mathbf{N}$.

Principle 2.6. Extended Second Principle of Mathematical Induction
Let $k \in \mathbf{N}$ and let S be a subset of $\mathbf{N}$ such that

a. $k \in S$, *and*

b. *if* $n \geq k$ *and* $\{k, k+1, ..., n\} \subset S$ *then* $n+1 \in S$.

Then $\{n \in \mathbf{N} | n \geq k\} \subset S$.

Principle 2.7. Well Ordering Principle or Least Natrual Number Principle
Every nonempty set of the natural numbers has a least member.

Theorem 2.8. *PMI* $\Leftrightarrow$ *LNNP* $\Leftrightarrow$ *SPMI*

Proposition 2.9. *Every natural number greater than 1 is either a prime number or the product of prime numbers. (Hint: SPMI)*

Theorem 2.10. The Division Algorithm for Integers
Let a be an integer and let b be a natural number. Then there exist integers q and r such that $a = bq + r$ and $0 \leq r < b$. Furthermore, the numbers q and r exist uniquely in the sense that if q' and r' are integers such that $a = bq' + r'$ and $0 \leq r < b$ then $q = q'$ and $r = r'$.

Proof uses LNNP.

Hints and group work for 3.1 Homework:

1. Hint for problem 3.1e
   (a) Consider the following sum of a sequence of numbers:
   $$2+5+8+...+(3n-1).$$
   What would be the next term (the $(n+1)^{st}$ term) of this sum?
   (b) Replace n by $n+1$ in the following expression
   $$\frac{n(3n+1)}{2}.$$
   (c) Use PMI to prove: For every natural number n,
   $$2+5+8+...+(3n-1)=\frac{n(3n+1)}{2}.$$
2. Hint for 3.5: Prove that the product of any three consecutive natural numbers is divisible by 6. When you get to $(n+1)(n+2)(n+3)$ remember that for any three numbers $AB(C+D)=ABC+ABD$. Applying this to $(n+1)(n+2)(n+3)$ should get you through the induction step.
3. Hint for problem 3.6
   (a) Find the sum of the first 4 odd numbers.
   (b) Find the sum of the first 5 odd numbers.
   (c) Find the sum of the first 6 odd numbers.
   (d) Make a conjecture (guess) as to what the sum of the first n odd numbers is.
   (e) Use PMI to prove your conjecture.
4. Hint of 3.8: Check carefully that the assumptions in the induction step apply to *all* $n \geq 1$.
5. Hint for problem 3.15: After you show that $1 \in S$. You will assume that $n \in S$ and want to show that $n+1 \in S$. As $n \in S$ we know that $(1+x)^n \geq 1+nx$. Now consider $(x+1)^{n+1}=(x+1)(x+1)^n$. Since $x>-1$ (by hypothesis) we have $x+1>0$ (why?). Use that and the fact $n \in S$ so show that $(x+1)(x+1)^n \geq (x+1)(1+nx)$ (why do you need $x+1>0$?). Now factor this out and use that $nx^2>0$ to finish the proof.

Group Work and Homework Hints for 3.2 Homework:

1. Prove that for each natural number $n \geq 6$ we have $2n + 3 \leq 2^n$.

2. (FP #20) Use the above to prove that for each natural number $n \geq 6$ we have $(n+1)^2 \leq 2^n$. [Hint: $n^2 + 4n + 4 = (n^2 + 2n + 1) + (2n + 3)$.]

3. (FP # 28) Consider the set $T = \{n \in \mathbf{N} | n > 1, n \text{ not prime and } n \text{ not the product of primes}\}$. You want to show that $T = \emptyset$ (why?). Well proceed by contradiction. Suppose that $T \neq \emptyset$. Then T has a least element (why?). Now call that least element d. Since $d \in T$ we know that:

   (a) $d \neq 1$,

   (b) d is not a prime, and

   (c) d is not a product of primes.

   The second of these properties shows that there exists $a, b \in \mathbf{Z}$ such that $d = ab$ where neither of a and b are equal to d nor are they equal to 1 (why?). Notice also that $a, b < d$ and thus they are NOT in T (why?). What does this tell you about a and b. From this info on a and b and what we know about d draw a contradiction.