# Course Notes for Math 320: Fundamentals of Mathematics Chapter 5: Functions.

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## 1 Introduction to Functions

**Definition 1.1.** 1. *A* **function** *is a (non-empty) relation $f$ such that if $a$ is in the domain of $f$ then there is ONE AND ONLY ONE ordered pair in the relation $f$ with first term equal to $a$. (Just to clarify, if $(a,b),(a,c) \in f$ then $b = c$.)*

2. *Notation: If the function $f$ is a relation from $X$ to $Y$ (i.e. $f \subset X \times Y$) then we write $f : X \to Y$ and if $(x,y) \in f$ then we write $f(x) = y$ or $x \mapsto y$ under $f$. Call the set $X$ the* **domain** *of $f$ and the set $Y$ the* **codomain** *of $f$.*

3. *The* **range** *or* **Image** *of a function $f$ is defined exactly as it is for relations:* $\{y \in Y | \exists x \in X \text{ with } f(x) = y\}$.

**Example 1.2.** *Which relations are functions?*

1. $X = \{1,2,3,4,5\}$ *and* $Y = \{a,b,c,d,e\}$. $f = \{(1,c),(2,b),(3,b),(4,d),(5,b)\}$. *What is* $f(3) =$?

2. $X = \{1,2,3,4,5\}$ *and* $Y = \{a,b,c,d,e\}$. $f = \{(1,c),(2,b),(3,a),(3,d),(5,b)\}$ . *What is* $f(3) =$?

3. $X = \{1,2,3,4,5\}$ *and* $Y = \{a,b,c,d,e\}$. *With the following graph:*

4. *$f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = x^2$. How is this a relation?*

5. *$f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3 + 1$. Where does $f$ take the interval $(0, 2)$?*

**Example 1.3.** *Let $P_n = \{f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \mid a_i \in \mathbf{R}\} \subset \mathbf{R}[x]$. This is the set of all polynomials of degree less than or equal to $n$.*

1. *Give some examples of elements in $P_2$.*

2. *Consider $D : P_3 \to P_2$ defined by $f(x) \mapsto f'(x)$. Find $D(2 + 3x + 5x^2 - x^3)$.*

3. *Find $D(a_0 + a_1x + a_2x^2 + a_3x^3)$.*

4. *What is the domain, codomain and image of $D$?*

5. *Find 5 functions* $g_i(x)$ *such that* $D(g_i(x)) = 3 + 10x - 3x^2$.

6. *Find all elements of the set* $\{f(x) \in P_3 \mid D(f(x)) = 3 + 10x - 3x^2\}$.

## 2 Injective and surjective

**Definition 2.1.** *Given a function* $f : X \to Y$

1. *We say that* $f$ *is* **surjective** *(or onto) if* $\forall\ y \in Y\ \exists\ x \in X$ *such that* $f(x) = y$.
2. *We say that* $f$ *is* **injective** *(or one to one) if* $\forall\ y \in Im(f)\ \exists$ *unique* $x \in X$ *such that* $f(x) = y$.

**Example 2.2.** *Are the following injective and/or surjective?*

1. $g : \mathbf{R}^2 \to \mathbf{R}^2$ *such that* $g(x, y) = (x + y, x - y)$.

2. $D : P_2 \to \mathbf{R}^3$ *such that* $D(f(x)) = (f(0), f'(0), f''(0))$.

*3.* $I : P_2 \to P_3$ *such that* $I(f(x)) = \int f(x)dx$.

*4.* $I : P_2 \to \mathbf{R}$ *such that* $I(f(x)) = \int_0^1 f(x)dx$.

## 3 Pre-Image, composition, inverses

**Definition 3.1.** *Given a function* $f : X \to Y$ *with* $S \subset Y$*. The* **preimage of** $S$ **with respect to** $f$ *is the set* $f^{-1}(S) = \{x \in X | f(x) \in S\}$*.*

**Example 3.2.** *Find the preimages:*

1. $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = \sin x$*. Find* $f^{-1}([0, 1])$*. Find* $f^{-1}(\{1\})$*.*

2. $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = x^2(x - 1)$*. Find* $f^{-1}([0, \infty])$*. Find* $f^{-1}(\{0\})$*.*

3. $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = x^3$*. Find* $f^{-1}([0, \infty])$*. Find* $f^{-1}(\{0\})$*.*

4. $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = e^x$*. Find* $f^{-1}([0, \infty])$*. Find* $f^{-1}(\{0\})$*.*

5. $D : P_3 \to P_2$ *defined by* $p(x) \mapsto p'(x)$*. Find* $D^{-1}(\{x^2\})$*.*

6. $Id_{P_2} : P_2 \to P_2$ *defined by* $p(x) \mapsto p(x)$. *Find* $Id_{P_2}(x^2)$. *Find* $Id_{P_2}^{-1}(\{x^2\})$.

7. $Id_U : U \to U$ *defined by* $u \mapsto u$. *Find* $Id_U(u)$. *Find* $Id_U^{-1}(\{u\})$.

**Remark 3.3.** 1. *Notice that* $f : X \to Y$ *is injective if and only if* $(\forall y \in Im(f))(\exists!\ x \in f^{-1}(\{y\}))$. *Proof?*

2. *Notice that* $f : X \to Y$ *is surjective if and only if* $(\forall y \in Y)(\exists\ x \in f^{-1}(\{y\}))$. *Proof?*

3. *Notice that* $f : X \to Y$ *is injective and surjective if and only if* $(\forall y \in Y)(\exists!\ x \in f^{-1}(\{y\}))$. *Proof?*

**Definition 3.4.** *Let* $f : X \to Y$ *be a function.*

1. *We call* $f$ **bijective** *(or a one-to-one correspondence) if it is injective and surjective.*

2. *If* $g : Y \to Z$ *is another function then* $g \circ f : X \to Z$ *is the function* $(g \circ f)(x) = g(f(x))$.

3. *The function* $f$ *is said to be* **invertible** *if there exists a function* $h : Y \to X$ *such that* $f \circ h = Id_Y$ *and* $h \circ f = Id_X$. *In this case* $h$ *is called the* **inverse** *of* $f$ *and we write* $h = f^{-1}$.

**Proposition 3.5.** *If* $f : X \to Y$ *is invertible then the inverse is unique.*

**Example 3.6.** *Which of the above is bijective? Invertible? What is it's inverse? Any conjectures about when an inverse exists?*

In our discussion of inverses, we used some properties of composite functions. Let's just spend another moment on these.

**Theorem 3.7.** *Let $f : X \to Y$ and $g : Y \to Z$ be functions.*

1. *If $f$ and $g$ are injective then so is $g \circ f$.*
2. *If $f$ and $g$ are surjective then so is $g \circ f$.*
3. *If $g \circ f$ is injective then so is $f$.*
4. *If $g \circ f$ is surjective then so is $g$.*

The absolute value function.

**Definition 3.8.** *The absolute value function is* $|\ |: \mathbf{R} \to \mathbf{R}$ *defined by* $|x| = max\{x, -x\}$.

**Theorem 3.9.** *Let* $p, q \in \mathbf{R}$. *The following hold:*

1. $|p - q| = 0$ *if and only if* $p = q$.
2. $|p - q| = |q - p|$
3. $|pq| = |p||q|$
4. $|p + q| \leq |p| + |q|$
5. $||p| - |q|| \leq |p - q|$
6. $\forall \epsilon > 0$, $|p| < \epsilon$ *if and only if* $-\epsilon < p < \epsilon$.

**Remark 3.10.** *Part (f) implies that the following are equivalent:*

1. $|x - a| < \epsilon$
2. $\quad < x - a <$
3. $a - \epsilon <$
4. $\quad < x <$

**Example 3.11.** *Find all* $x \in \mathbf{R}$ *such that* $|x - 1| < \frac{1}{5}$.

**Definition 3.12.** *A bijection $f : X \to X$ is called a* **permutation** *of $X$. It is denoted $Perm(X)$.*

**Remark 3.13.**

1. *If $\sigma, \tau \in Perm(x)$ then $\sigma \circ \tau$ and $\tau \circ \sigma$ both make sense. Why?*
2. *If $\sigma, \tau \in Perm(x)$ then $\sigma \circ \tau, \tau \circ \sigma \in Perm(x)$. Why?*
3. *If $\sigma \in Perm(x)$ then its inverse $\sigma^{-1}$ exists. Why?*
4. *If $\sigma \in Perm(x)$ then its inverse $\sigma^{-1} \in Perm(X)$. Why?*
5. *If $\sigma, \tau, \gamma \in Perm(X)$ then $(\sigma \circ \tau) \circ \gamma =$?*
6. *Let $X$ be a triangle. Compute $Perm(X)$.*

**Definition 3.14.** *A set $G$ together with a binary operation $*$ on $G$ is called a* **group** *if the following all hold:*

1. *$*$ is associative.*
2. *there exists an identity element for $*$.*
3. *every $g \in G$ has an inverse element in $G$ with respect to $*$.*

# 4 Homework

1. Which of the following functions are one to one and which are onto? WHY?
   (a) $f(x) = 3x - 5$
   (b) $f(x) = x^2 + 5$
   (c) $f(x) = x^2 - 5x - 6$
   (d) $f(x) = x^3 - 5$
   (e) $f(x) = x^3 - x$.
2. F & P Chapter 5 # 4.
3. F & P Chapter 5 # 7.
4. F & P # 73.
5. Let $M_2(\mathbf{R})$ be the set of $2 \times 2$ matrices with real entries. That is:

$$M_2(\mathbf{R}) = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} | a_{11}, a_{12}, a_{21}, a_{22} \in \mathbf{R} \right\}.$$

   Define $Tr : M_2(\mathbf{R}) \to \mathbf{R}$ by $Tr(A) = a_{11} + a_{22}$. This is called the trace of the matrix. Is the trace function injective? Is it surjective? Support your answers with proofs or counter examples.
6. Let $M_2(\mathbf{R})$ be as above. Let $Det : M_2(\mathbf{R}) \to \mathbf{R}$ be defined by $Det(A) = a_{11}a_{22} - a_{21}a_{12}$. This is called the determinant of the matrix $X$. Is the function $Det$ injective? Is it surjective? Support your answers with proofs or counter examples.
7. Let $M_2(\mathbf{R})$ be as above and let $P_2$ be polynomials of degree less than or equal to 2. Let $Char : M_2(\mathbf{R}) \to P_2$ be defined by $A \mapsto x^2 - Tr(A)x + Det(A)$. Is the function $Char$ injective? Is it surjective? Support your answers with proofs or counter examples. [Hint: Think of matrices that have both determinant and trace equal to zero.]
8. Prove the following theorem:

   **Theorem 4.1.** *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ is injective then so is $f$.*