

# Course Notes for Math 320: Fundamentals of Mathematics Combinatorics.

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## 1 Introduction to Combinatorics

### Typical combinatorial questions:

1. Given a set of objects, does a certain arrangement of those objects exist?
2. How many such arrangements exist?
3. What is the probability of getting this arrangement?
4. What properties does this arrangement have?
5. Which is the optimal arrangement given certain criteria?

### Typical techniques for answering combinatorial questions:

1. The Pigeonhole principle
2. The inclusion/exclusion principle
3. Various counting techniques
4. Probability
5. Analysis
6. Ad hoc arguments!!!

**Example 1.1.** *Name tags: If A has B's name take then A should hold B's lefthand with his or her righthand. GO!*

1. *How many ways are there to pass out  $n$  name tags to  $n$  people?*

2. *How many of these include someone getting his or her own nametag?*

3. *How many ways are there so that no one gets his or her own nametag?*

4. *What configurations relate to the previous two questions?*

5. *How many possibilities give exactly 5 circles?*

6. *How many possibilities have a circle of size 5?*

7. *How many possibilities give  $r$  circles of size  $n_1, n_2, \dots, n_r$ ? Condition on  $n_i$ ?*

**Example 1.2.** *More combinatorial questions:*

1. *How many sequences  $a_1, a_2, \dots, a_{12}$  are there consisting of four 0's and eight 1's if no two consecutive terms are both zero?*

2. *An ice-cream parlor serves 10 flavors of icecream.*

*(a) A medium bowl holds 3 scoops: How many distinct bowls are possible?*

*(b) A medium cone holds 3 scoops: How many distinct cones are possible?*

*(c) What is the difference?*

3. *A bakery has 8 kinds of bagles. A bag holds a dozen bagles. How many distinct bags can be formed with at least one bagle of each kind?*

4. *CSUN college basketball team has 12 players. In practice they play 5 on 5 with red and blue jerseys. How many configurations are possible?*

## 2 Counting I: Addition and Multiplication Principles.

**Example 2.1.** *Warm-up: You have one orange and eight apples. You want to give President Koester a non-empty basket of fruit. How many different possibilities are there?*

**Definition 2.2.** *Recall that a **partition** of a set  $S$  is a collection of subsets  $S_i \subset S$  such that  $\cup_i S_i = S$  and  $S_i \cap S_j = \emptyset$ .*

**Theorem 2.3.** *(The addition principle) If  $S$  is partitioned into subsets  $S_1, \dots, S_r$  then  $|S| = |S_1| + |S_2| + \dots + |S_r|$ .*

**Remark 2.4.**

1. *Addition principle  $\Leftrightarrow$  Cases.*
2. *Art of using Addition Principle:*
3.  $\exists$  *more interesting version when  $S_i \cap S_j \neq \emptyset$ . (later)*

**Theorem 2.5.** *(The multiplication Principle) Let  $S$  and  $T$  be sets containing  $p$  and  $q$  elements each. Then the set  $S \times T$  has  $pq$  elements.*

**Example 2.6.** *(TASKS) How many two digit numbers have distinct non-zero digits?*

**Example 2.7.** *How many odd numbers are there between 1000 and 9999? How many with distinct digits?*

**Example 2.8.** *How many numbers between 1 and 10,000 have exactly one 5?*

**Example 2.9.** *A bookshelf holds 6 different books in English, 8 different books in Spanish, 10 different books in German, and 2 different books in Japanese.*

1. *How many ways are there of selecting 4 books in such a way that you have one in each language?*

2. *How many ways are there of selecting one book in any language?*

**Remark 2.10.** *General Idea:*

*If you divide into mutually exclusive CASES then you ADD.*

*If you divide into sequential TASKS then you MULTIPLY.*

**Definition 2.11.** Let  $s, t \in \mathbf{N}$ .

1. A  $t$ -set is a set  $T$  with  $t$  (distinct) elements.
2. For  $s \leq t$  an  $s$ -permutation of a  $t$ -set  $T$  is a sequence (i.e. ordered) of  $s$  of the elements of  $T$ .
3. For  $s \leq t$  an  $s$ -combination of a  $t$ -set  $T$  is a set (i.e. unordered) of  $s$  of the elements of  $T$ .

**Remark 2.12.** A permutation of  $T$  is a  $t$ -permutation of  $T$ .

**Example 2.13.** Back to icecream: We can choose from 10 flavors of icecream. Which is a 3-permutation? a 3-combination? Neither?

1. Triple scoop cone with distinct flavors.
2. Triple scoop cone.
3. Triple scoop bowl.
4. Triple scoop bowl with distinct flavors.

**Theorem 2.14.** *Let  $s, t \in \mathbf{N}$  with  $s \leq t$ . The following are all equal numbers:*

1. *The number of ways of placing  $s$  distinct balls into  $t$  distinct urns so that no urn has more than one ball.*
2. *The number  $[t]_s$  of  $s$ -permutations of a  $t$ -set  $T$ .*
3. *The number  $\frac{t!}{(t-s)!}$ .*

**Theorem 2.15.** *Let  $s, t \in \mathbf{N}$  with  $s \leq t$ . The following are all equal numbers:*

1. *The number of ways of placing  $s$  identical balls into  $t$  distinct urns so that no urn has more than one ball.*
2. *The number  $\binom{t}{s}$  of  $s$ -combinations of a  $t$ -set  $T$ .*
3. *The number  $\frac{t!}{s!(t-s)!}$ .*

**Theorem 2.16.** *Identities for  $\binom{n}{k}$ : Let  $n, k \in \mathbf{Z}_{\geq 0}$ .*

1.  $\binom{n}{k} = \binom{n}{n-k}$

2.  $\sum_{k=0}^n \binom{n}{k} = 2^n$

3.  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

4.  $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

### 3 Pigeonhole Principle

**Example 3.1.** *Warm-up: When is the following statement true? In a group of  $n$  people at least two have a birthday in the same month.*

**Theorem 3.2.** *(Pigeonhole Principle)*

1. *If we put  $n + 1$  pigeons into  $n$  pigeonholes then at least one pigeonhole has more than one pigeon.*
2. *Let  $R$  and  $D$  be finite sets and  $f : D \rightarrow R$  be a function. If  $|D| > |R|$  then  $f$  is **not** injective (one-to-one).*

**Remark 3.3.** *On Pigeons:*

1. *To relate parts (a) and (b): Let  $D$  be the set of pigeons and let  $R$  be the set of pigeonholes. Then  $f : R \rightarrow D$  is defined by???*
2. *In example above: Let  $D$  be the set of people and  $R$  be the months of the year. Then  $f : D \rightarrow R$  is defined by assigning to each person the month of his or her birthday.*

**Example 3.4.** *(Medium example) Five points are placed anywhere in the interior of a unit square. Prove that two of them lie at distance strictly less than  $\frac{\sqrt{2}}{2}$  apart.*

**Example 3.5.** (*Medium-hard example*) At a party some people present shook hands with some of the others. Assume that at least two people were at the party. Show that there exists at least two people who shook hands with the same number of people.

**Example 3.6.** (*Hard example*) A chess master has 11 weeks to prepare for a tournament. She decides to play at least one game every day and to play no more than 12 games per week. Show that there exists a succession of days during which she plays exactly 21 games.

**Definition 3.7.** Let  $\langle b_1, b_2, \dots, b_r \rangle$  be a sequence of real numbers.

1. A subsequence of  $\langle b_1, b_2, \dots, b_r \rangle$  is a sequence  $\langle b_{i_1}, b_{i_2}, \dots, b_{i_s} \rangle$  such that  $b_{i_j} = b_k$  (i.e.  $i_j = k$ ) for some  $k \in \{1, 2, \dots, r\}$  and  $i_1 < i_2 < \dots < i_s$ .
2. A subsequence  $\langle b_{i_1}, b_{i_2}, \dots, b_{i_s} \rangle$  is said to be increasing if  $b_{i_1} \leq b_{i_2} \leq \dots \leq b_{i_s}$ .
3. A subsequence  $\langle b_{i_1}, b_{i_2}, \dots, b_{i_s} \rangle$  is said to be decreasing if  $b_{i_1} \geq b_{i_2} \geq \dots \geq b_{i_s}$ .

**Example 3.8.** Consider the sequence  $\langle 3, 5, 2, 8, 5, 6, 9 \rangle$

1. Find an increasing subsequence.
2. Find a decreasing subsequence.
3. What is  $\langle 5, 5 \rangle$ .

**Theorem 3.9.** (Erdős-Szekeres) Let  $n$  be a positive integer. Any sequence of  $n^2 + 1$  real numbers contains either an increasing or decreasing subsequence of  $n + 1$  real numbers.

**Example 3.10.** (Warm-up)

1. Take  $n = 3$  so  $n^2 + 1 = 10$ . In any sequence of 10 people there is a subsequence of 4 people of increasing or decreasing height.
2. Take  $n = 2$  so  $n^2 + 1 = 5$  consider  $\langle 2, 5, 1, 3, 4 \rangle$ .
  - (a) Find the longest increasing subsequence starting with 2.
  - (b) Find the longest decreasing subsequence starting with 2.
  - (c) Find the longest increasing subsequence starting with 5.
  - (d) Find the longest decreasing subsequence starting with 5.
  - (e) Find the longest increasing subsequence starting with 1.
  - (f) Find the longest decreasing subsequence starting with 1.
  - (g) Find the longest increasing subsequence starting with 3.
  - (h) Find the longest decreasing subsequence starting with 3.
  - (i) Find the longest increasing subsequence starting with 4.
  - (j) Find the longest decreasing subsequence starting with 4.