Course Notes for Math 320: Fundamentals of Mathematics Chapter 1: Generalities on proofs.

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1 Propositions, Truth

Definition 1.1 *A* **proposition** *is a sentence that is either true or false.*

Example 1.2 *Which one is a proposition?*

1. *This summer I spent a month in Cortina, Italy.*
2. *Are Nilla wafers your favorite cookie?.*
3. *Nilla wafers are the best cookies.*
4. $5^2 + 3^2 = 34$.
5. $5^2 + 3^2 = 35$.
6. $5^2 + 3^2 < 35$.
7. $x + 3$.
8. $x + 3 > 5$.
9. *If ____________ is registered for math 320 then ____________ must have received a C or better in math 150B (or its equivalent).*

Making new propositions out of old ones: Negation, conjunction (and), disjunction (or), implication (if, then).

Definition 1.3 *Given a proposition P, the further proposition that P is false is called the* **negation** *of P (or Not$P$).*

Example 1.4 *Negate the following:*

1. *This summer I spent a month in Cortina, Italy.*

2. $5^2 + 3^2 = 34$.

3. $5^2 + 3^2 = 35$.

4. $5^2 + 3^2 < 35$.

5. *If Sam is registered for math 320 then he must have received a C or better in math 150B (or its equivalent).*

Notice that if the statement P is true, then the statement Not$P$ is false and if the statement P is false then the statement Not$P$ is true. We summarize this in a "Truth Table:"

P	Not P
T	F
F	T

Definition 1.5 *A proposition of the form "P and Q" for two proposition P and Q is called the* **conjunction** *of P and Q. Notation:* $P \cap Q$*.*

Definition 1.6 *A proposition of the form "P or Q" for two proposition P and Q is called the* **disjunction** *of P and Q. Notation:* $P \cup Q$*.*

Example 1.7 *Which of the following are true?*

1. *This summer I spent a month in Cortina, Italy and* $5^2 + 3^2 = 34$.

2. *This summer I spent a month in Cortina, Italy and* $5^2 + 3^2 = 35$.

3. $5^2 + 3^2 = 35$ *and* $5^2 + 3^2 = 34$.

4. $5^2 + 3^2 = 35$ *and* $5^2 + 3^2 > 35$.

Example 1.8 *Which of the following are true?*

1. *This summer I spent a month in Cortina, Italy or* $5^2 + 3^2 = 34$.

2. *This summer I spent a month in Cortina, Italy or* $5^2 + 3^2 = 35$.

3. $5^2 + 3^2 = 35$ *or* $5^2 + 3^2 = 34$.

4. $5^2 + 3^2 = 35$ *or* $5^2 + 3^2 > 35$.

So "P and Q" is true only when *both* P and Q are true, and "P or Q" is true when either P or Q is true. Again this can be summarized a truth table.

P	Q	P and Q	P or Q
T	T	T	
T	F		T
F	T		T
F	F	F	

Definition 1.9 *Given two propositions P and Q, the proposition "P* **implies** *Q" is called a* **conditional** *proposition. Notation $P \Rightarrow Q$.*

Example 1.10

1. *If Giuseppe is nice to Matteo then he will have an icecream.*
2. *Giuseppe is in his carseat implies he is dressed.*
   Or equivalently, if Giuseppe is in his carseat then he is dressed.
3. *If Giuseppe is dressed then he is in his carseat.*
4. *If Sam is allowed to register for math 320 then he received a C or better in math 150B (or its equivalent).*

Thus the truth or falsehood of the expression "P implies Q" is defined by the following truth table.

P	Q	P implies Q
T	T	T
T	F	F
F	T	T
F	F	T

In this case P is called the **hypothesis** and Q is called the **conclusion**. See page 5 of text for various other equivalent expressions: e.g. "If P, then Q" or "Q is true whenever P is true."

Example 1.11 *Which of the following are true?*

1. *If I spent a month in Cortina, Italy this summer, then* $5^2 + 3^2 = 34$.
2. *If I spent a month in Cortina, Italy this summer, then* $5^2 + 3^2 = 35$.
3. *If* $5^2 + 3^2 = 35$, *then* $5^2 + 3^2 = 34$.

4. *If* $5 + 3 = 7$, *then* $5^2 + 3^2 > 35$.

Example 1.12 *Mount Holyoke College (MHC) is a women's college. Assume Mary is a woman and a student at MHC. Assume also that Sam is not a student at MHC and that Sam is a man.*

1. *If Mary is a student at MHC then Mary is a woman.*
2. *If Sam is a student at MHC then Sam is a man.*
3. *If a student is a woman, then the student attends MHC.*

Definition 1.13 *Given two propositions P and Q, the* **converse** *to the proposition "P implies Q" is "Q implies P."*

Example 1.14 *Write the converse to the following. Which do your intuition tell you are true?*

1. *If Giuseppe is in his carseat then he is dressed.*
2. *To register for math 320, it is necessary that a student received a C or better in math 150B (or its equivalent).*
3. *If you are over 21 years old you may legally drink alcohol.*

Thus the truth or falsehood of the converse to the expression "P implies Q" is defined by the following truth table.

P	Q	P implies Q	Q implies P
T	T	T	
T	F	F	
F	T	T	
F	F	T	

Definition 1.15 *Given two propositions P and Q, the proposition "P* **if and only if** *Q" is true when both "P implies Q" and "Q implies P" are true. Notation $P \Leftrightarrow Q$.*

Example 1.16 *Which do your intuition tell you are true?*

1. *Giuseppe is in his carseat $\Leftrightarrow$ he is dressed.*

2. *A student takes math 320 if and only if the student received a C or better in math 150B (or its equivalent).*

3. *You may legally drink alcohol if and only if you are over 21 years old.*

Example 1.17 *Which of the following are true?*

1. *I spent a month in Cortina, Italy this summer* $\Leftrightarrow 5^2 + 3^2 = 34$.
2. *I spent a month in Cortina, Italy this summer if and only if* $5^2 + 3^2 = 35$.
3. $5^2 + 3^2 = 35$ *if and only if* $5^2 + 3^2 = 34$.
4. $5 + 3 = 7$ *if and only if* $5^2 + 3^2 > 35$.

Definition 1.18 *Given two propositions P and Q, the* **contrapositive** *to "P* **implies** *Q" is the proposition "NOTQ implies NOTP."*

Example 1.19 *Find the contrapositive for each of the following. Which are true?*

1. *If Giuseppe is in his carseat then he is dressed.*
2. *If Sam is allowed to register for math 320 then he received a C or better in math 150B (or its equivalent).*
3. *If I spent a month in Cortina, Italy this summer, then* $5^2 + 3^2 = 34$.
4. *If I spent a month in Cortina, Italy this summer, then* $5^2 + 3^2 = 35$.
5. *If* $5^2 + 3^2 = 35$*, then* $5^2 + 3^2 = 34$.
6. *If* $5 + 3 = 7$*, then* $5^2 + 3^2 > 35$.

P	Q	P implies Q	Not Q	Not P	$(Not\ Q \Rightarrow Not\ P)$	$(P \Rightarrow Q) \Leftrightarrow (Not\ Q \Rightarrow Not\ P)$
T	T	T	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

Definition 1.20 *A propositional expression is called a* **tautology** *if it yeilds a true proposition regardless of what propositions replace its variable expressions.*
A propositional expression is called a **contradiction** *if it yeilds a false proposition regardless of what propositions replace its variable expressions.*

Example 1.21 $(P \Rightarrow Q) \Leftrightarrow (Not\ Q \Rightarrow Not\ P)$. *We say that these expressions are logically equivalent.*

This can be useful when proving an if and only if statement, like

Example 1.22 *"You may legally drink if and only if you are over 21 years old."*

1. $(Q \Rightarrow P)$

2. $(Not\ Q \Rightarrow Not\ P)$;

Sometimes rephrasing things makes the truth or falsity of a statement jumps out. A few more useful logical equivalences (or tautologies) are De Morgan's Laws.

Proposition 1.23 *Each of the following propositional expressions is a tautology.*

1. $Not(P \Rightarrow Q) \Leftrightarrow P \cap Not\ Q$.
2. $Not(P \cup Q) \Leftrightarrow Not\ P \cap Not\ Q$.
3. $Not(P \cap Q) \Leftrightarrow Not\ P \cup Not\ Q$.
4. $Not(Not\ P) \Leftrightarrow P$.
5. $(P \cup Q) \Leftrightarrow (NOT\ P \Rightarrow Q)$.

These are mostly of use in negating complicated expressions.

Example 1.24 *Negate the following:*

1. *If Giuseppe is in his carseat then he is dressed.*
2. *If* $5 + 3 = 8$ *then* $f(5) + f(3) = f(8)$.
3. *If* $ab = 0$ *then* $a = 0$ *or* $b = 0$.

2 Constants, Variables, and Quantifiers

Two types of objects: **constants** and **variables**.
Both live in some **universe**... that will have to be defined in each situation.

Example 2.1 $x + 3 > 5$
This is not a propostion.
Here 3 *and* 5 *are a constants,* x *is a variable.*
Where does x *come from? Which "universe" of numbers? Real numbers (*$\mathbf{R}$*), the rational numbers (*$\mathbf{Q}$*) or just integers (*$\mathbf{Z}$*) .*

Example 2.2 $x + 3 > 5$ *for some real number* x*. Is this a proposition? Is it true?*

Definition 2.3 *Given a universe and a variable* x*, a* **Propositional function** $P(x)$ *in the variable* x *is a sentence such that for each value of* x *taken in the universe we obtain a proposition. The* **truth set** *of* $P(x)$ *is the set of all values* x *in the given universe such that* $P(x)$ *is a true proposition.*

Example 2.4 *Find the truth set of each of the following propositional functions. Assume that all variables and constants are real number unless otherwise stated.*

1. $x + 3 > 5$
2. $x^2 + y^2 = 4$
3. $x^3 + y^3 = z^3$ *where none of* x*,* y *and* z *are zero.*
4. *Let the universe be the set of all CSUN. Let* X *be one such student,* X *got a C or better in math 150B.*

Notation:

Quantifiers to Propositional functions:

Example 2.5 *Are the following* **Propositions** *true or false?*

1. **There exist** *real numbers* x *such that* $x + 3 > 5$
2. **For all** *pairs of real number* (x, y)*,* $x^2 + y^2 = 4$*.*

3. **There are no** *integer solutions to* $x^3 + y^3 = z^3$ *where none of* x, y *and* z *are zero.*

4. *Let the universe be the set of all CSUN.* **There exists** X *who got a* C *or better in math 150B.*

Caution: What is the difference between "there exists" and "for all?"

Example 2.6 *What are the differences between the following propositions?*

1. *All students in math 320 get an* A.

2. *There exists a student in math 320 who gets an* A.

3. *For all real numbers* x, $x^2 > 0$.

4. *There exists a real number* x, *such that* $x^2 > 0$.

   For 5-8, recall that a prime number is a positive integer $p > 1$ such that the only positive integers that divide it are 1 and p. For example $2, 3, 5, 7$ and 11 are primes, but $9 = (3)(3)$ is not.

5. *For any prime numbers* p, $p + 2$ *is a prime number.*

6. *For some prime numbers* p, $p + 2$ *is a prime number.*

7. *There exists a prime number* p *such that* $p + 1$ *is prime.*

8. *There exists an odd prime number* p *such that* $p + 1$ *is prime.*

Definition 2.7 *Quantifiers*

1. *In a proposition any expression that indicates the existence of at least one element lying in the truth set of a propositional funtion is called an* **existential quantifier** *and it can be represented by* $\exists$. **(Rmk:** $\exists x \in U$ **is always followed by the expression (maybe just implied) "such that".)**

2. *Any expression that indicates that all elements in the given universe lie in the truth set of a propositional function is called a* **universal quantifier** *and can be represented by* $\forall$.

Example 2.8 *Translate either into the* $\forall$ *or* $\exists$ *notation or out of it. Which are true? For those that are false state the negation.*

1. $\forall x \in \mathbf{R}, x^4 > x$.

2. *Let* $\mathbf{P}$ *be the set of all prime numbers.* $\forall p \in \mathbf{P}$, $p + 2 \in \mathbf{P}$.

3. *For some prime numbers p, $p+2$ is a prime number.*

4. *There exists a prime number p such that $p+1$ is prime.*

Example 2.9 *The quantifiers may be hidden and these propositions can be more complicated. Two of the statements below are false... which ones, and can you prove it?*

1. *If x is a real number then $e^x > x$.*
2. *For a positive integer to be prime it is sufficient that it is odd.*
3. *Some Mammals can fly.*
4. *Some mammals have gills.*
5. *(Continuity of the function e^x at $x = 2$.) $\forall \epsilon > 0, \exists \delta > 0$, such that if $|x-2| < \delta$ then $|e^x - e^2| < \epsilon$.*
6. *Given a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n \in \mathbf{R}^n$. $\forall \vec{v} \in \text{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$, $\exists$ unique $a_1, ..., a_n \in \mathbf{R}$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$.*

Proposition 2.10 *If $P(x)$ is a propositional function with variable x, then the following indicates how to negate an $\forall$ or $\exists$ expression.*

1. $Not\,((\forall x)P(x)) \Leftrightarrow (\exists x)\,(Not P(x))$ *(counter-example);*
2. $Not\,((\exists x)P(x)) \Leftrightarrow (\forall x)\,(Not P(x))$.

Example 2.11 *Negate the following:*

1. *For some prime numbers p, $p+2$ is a prime number.*

2. *Let $\mathbf{P}$ be the set of all prime numbers. $\forall p \in \mathbf{P}$, $p+2 \in \mathbf{P}$.*

Quantifiers and Defintions.

Recall (?) the following definitions:

Def: A function f is said to be continuous at a value $x = x_0$ if $\forall \epsilon > 0, \exists \delta > 0$, such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

Def: Given a set of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n \in \mathbf{R}^n$, another vector $\vec{v}$ lies in $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ if there exist $a_1, a_2, ..., a_n \in \mathbf{R}$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$.

Example 2.12 *How propositions and definitions interact.*

1. *What would you have to show in order to say that a function $g(x)$ is continuous at a point x_0?*

2. *What would you have to show in order to say that a vector $\vec{v}$ lies in the* $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$*?*

3. *State what it would mean for a function $g(x)$ to fail to be continuous at a point x_0.*

4. *State what it would mean a vector $\vec{v}$ to fail to lie in the* $\mathrm{Span}(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$*?*

3 Direct, Contrapositive, Contradition Proofs

Direct Proof of ($P \Rightarrow Q$)	Contrapositive Proof ($P \Rightarrow Q$)	Contradiction Proof ($P \Rightarrow Q$)
Assume P	Assume $Not\ Q$	Assume P and $Not\ Q$
Logical sequence of steps	Logical sequence of steps	Logical sequence of steps
Conclude Q.	Conclude $Not\ P$	Conclude Propositions R and $Not\ R$

Theorem 3.1 *An integer is even if and only if n^2 is even.*

4 Group Work:

Definition 4.1 *Let a and b be integers.*

1. *We say that a divides b if $b = ac$ for some positive integer c. We sometimes denote this by $a|b$.*
2. *We say that b is even if there is an integer k such that $b = 2k$ (i.e. 2 divides b).*
3. *We say that b is odd if there is an integer k such that $b = 2k + 1$.*

From lecture and book:

Theorem 4.2 *An integer is even if and only if n^2 is even.*

Warm-up: Prove the following:

1. Lemma 1: Let m and n be odd integers, then mn is an odd integer.
   Hint: write m and n out as odd integers using the definition above.
2. Lemma 2: If b is odd then b^3 is odd.
3. Lemma 3: If b^3 is even then b is even.

Group Exercise:

1. (FP 1.74) In this exercise we give both a "proof" and a "counterexample" to the conjecture: If x, y, and z are natural numbers such that xz divides yz, then x divides y. Is the "proof" correct? If not, why not? Is the "counter-example correct? If not, why not?

   "Proof:" Suppose x, y, and z are natural number such that xz divides yz but x does not divide y. Then y divides x, so there is an integer p such that $x = py$. Thus

$xz = (py)z = p(yz)$. There fore yz divides xz, and this is a contradiction because xz divides yz.

"Counter-example:" Let $x = 4$, $y = 6$ and $z = 16$. Then xz and divides yz but x does not divide y.

2. (FP1.73) Let a and b be positive integers such that $a^2 = b^3$. Come up with some examples.

   (a) Prove that: Given $a, b \in \mathbf{Z}$ if $a^2 = b^3$ and a is even, then 4 divides a.
   Hint:

      i. Assume that a is even and write $a = 2m$
      ii. Use the above theorem to show that a^2 is even.
      iii. Use this to show that b^3 is even.
      iv. Use this and the warm-up to show that b is even and write that $b = 2n$.
      v. Use this to show that this implies that 8 divides b^3.
      vi. Justify the equality $8n^3 = 4m^2$.
      vii. Use this to show that m^2 is even.
      viii. Use this and the warm-up to show that m is even.
      ix. Now show that this implies that 4 divides a.

   (b) Given that 4 divides a and that $a^2 = b^3$, prove that 4 divides b.
   Hint:

      i. Write $a = 4k$. What is $b^3 = a^2$ in terms of k? Show that b^3 is even.
      ii. Use the warm-up to show that b is even and write $b = 2c$.
      iii. Justify the expression $8c^3 = 16k^2$.
      iv. Show that c^3 is even and conclude (how?) that c is even.
      v. Use this to see that 4 divides b.

   (c) Given that 4 divides b and $a^2 = b^3$, prove that 8 divides a. Hint:

      i. Write $b = 4e$. What is b^3 in terms of e?
      ii. Justify the expression $a^2 = 4^3e^3$.
      iii. Use the theorem to say that a is even and write $a = 2z$.
      iv. Justify the following $4z^2 = 4^3e^3$.
      v. Use this to show that z is even, and let $z = 2q$.
      vi. Justify $4q^2 = 16e^3$.
      vii. Use this to show that q is even and write $q = 2d$.
      viii. Now $a = 2z = \_\_\_q = \_\_\_\_d$.

   (d) Give an example of positive integers a and b such that a is even and $a^2 = b^3$, but b is not divisible by 8.

5 Projects for presentation for this Chapter

There will be three presentations on the third Friday of the term. Each will be 15 minutes long. Here are some suggestions: Fletcher & Patty Chapter 1, # 60, 67, 73(a) and/or (b) and/or (c) and (d), 86, 90. Section 5. Exercise 86, Exercise 90 (a) or (b), and Exercise 91 (a) or (b).