Course Notes for Math 320: Fundamentals of Mathematics Analysis.

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1 Completeness Property

Definition 1.1. *Boundedness: Let $A \subset \mathbf{R}$.*

1. *A is* **bounded above** *if there is a real number $b \in \mathbf{R}$ such that $a \leq b$ for all $a \in A$.*
2. *A is* **bounded below** *if there is a real number $l \in \mathbf{R}$ such that $a \geq l$ for all $a \in A$.*
3. *$s \in \mathbf{R}$ is called the* **least upper bound** *of A if it is an upper bound and for any other upper bound b we have $b \geq l$. Notation: $s = lub(A)$.*
4. *$t \in \mathbf{R}$ is called the* **greatest lower bound** *of A if it is an lower bound and for any other lower bound l we have $l \leq t$.*
   Notation: $t = glb(A)$.

Lemma 1.2. *The $lub(A)$ and the $glb(A)$ are unique.*

Example 1.3. *1.* $A = \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\}$.

2. $B = (0, 1)$.

3. $C = (-2, \infty)$.

4. $D = \{r \in \mathbf{Q} \mid r^2 < 2\}$

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n	n^2	*upper bound?*
1.42	2.0164	*yes*
1.415	2.002225	
1.4145	*2.00081025*	
1.4144	*2.00052736*	
1.41422	*2.000018208*	

Completeness Axiom *Every non-empty subset of the real numbers which is bounded above has a least upper bound*

Theorem 1.4. *(Archimedean Property)*

1. *Given any* $x \in \mathbf{R}_{>0}$*, there exists an* $n \in \mathbf{N}$ *such that* $n > x$*.*
2. *Given any* $y \in \mathbf{R}_{>0}$*, there exists an* $n \in \mathbf{N}$ *such that* $\frac{1}{n} < y$*.*

2 Sequences

Definition 2.1. *A* **sequence** *is a function* $f : \mathbf{N} \to \mathbf{R}$.

Remark 2.2. *Since* $\mathbf{N}$ *is ordered then we can order the elements of a sequence:*

$$f(1) = a_1, f(2) = a_2, ..., f(n) = a_n, ...$$

In fact, giving this "list" is enough to determine the sequence and we usually write

$$(a_n) = \{a_1, a_2, ..., a_n, ...\}$$

Example 2.3. *Sequences.*

1. $\left(\frac{1}{n}\right) =$
2. $\left(\frac{n-1}{n}\right) =$
3. $(2^n) =$
4. $\left(1 + \frac{(-1)^n}{n}\right) =$
5. (a_n) *where* $a_1 = 2$ *and* $a_{n+1} = \frac{a_n+1}{2}$ *(this is called a recursively defined sequence).*

6. $\left(\sin(\frac{1}{n})\right) =$

Remark 2.4. *Basic question: What is the "long term" behavior? What happens to* a_n *as* $n \to \infty$*?*

Example 2.5. *What is the long term behavior of these sequences?*

1. $\left(\frac{1}{n}\right)$
2. $\left(\frac{n-1}{n}\right)$
3. (2^n)
4. $\left(1 + \frac{(-1)^n}{n}\right)$
5. (a_n) *where* $a_1 = 2$ *and* $a_{n+1} = \frac{a_n+1}{2}$ *(this is called a recursively defined sequence).*

6. $\left(\sin(\frac{1}{n})\right)$. *Recall that* $\sin x \leq x$.

Remark 2.6. *Building the definition of a convergence:*

1. *Intuitive Definition of limit:* $\lim a_n = A$ *means ...*

2. *Translate the following statement into "mathy language": The distance between* a_n *and* A *is less than* $\frac{1}{2}$.

3. *Translate the following statement into "mathy language": The terms of the sequence* (a_n) *from some point on, are within* $\frac{1}{2}$ *of* A.

4. *If $a_n \to A$ then we want the above to hold for any value $\epsilon > 0$. Translate the following statement into "mathy language": For any $\epsilon > 0$, the terms of the sequence (a_n) from some point on, are within ϵ of A.*

Definition 2.7. *Given a sequence (a_n) and $A \in \mathbf{R}$ we say that (a_n)* **converges to** *A if $\forall \epsilon > 0$ there is an $M \in \mathbf{N}$ such that $\forall n \geq M$ we have $|a_n - A| < \epsilon$. If no such A exists, then we say that (a_n)* **diverges**.

Example 2.8. *What is the long term behavior of these sequences?*

1. $\left(\frac{1}{n}\right)$
2. $\left(\frac{n-1}{n}\right)$
3. (2^n)

3 Homework I on Analysis

1. Consider $A = [2, 4)$. Show that 5 is an upper bound but not a least upper bound for A. Prove that 2.1 is not a lower bound for A and that 4 is the least upper bound for A.

2. Show that 5 is an upper bound but not a least upper bound for the set $S = \{x \in \mathbf{R} \mid \exists n \in \mathbf{N} \textit{ such that } x = 1 - \frac{1}{n}\}$. Find its least upper bound.

3. Follow the steps below to prove the following theorem: (Density of $\mathbf{Q}$ in $\mathbf{R}$) For any $a, b \in \mathbf{R}$ with $a < b$ there is a rational number $r \in \mathbf{Q}$ such that $a < r < b$.

   (a) First we will prove the case that $a \geq 0$. So, assume that $a, b \in \mathbf{R}$ with $a < b$ and $a \geq 0$. Write that down.

   (b) Explain why there is an $n \in \mathbf{N}$ such that $\frac{1}{n} < b - a$.

   (c) Show that then $a < b - \frac{1}{n}$.

   (d) Let m be the smallest natural number such that $m > na$ (why does it exist?).

   (e) Prove that $m - 1 \leq na < m$ and $a < \frac{m}{n}$.

   (f) Prove that $m \leq na + 1$.

   (g) Use this and part (c) to show that $m < n(b - \frac{1}{n}) + 1$.

   (h) Show that this implies that $m < nb$ and thus $\frac{m}{n} < b$.

   (i) Thus $a < \frac{m}{n} < b$.

   (j) Now if $a < b < 0$, then observe that $-a > -b > 0$. Use the above to finish the argument.

4. Follow the steps below to prove the following theorem: (Existence of square roots) There exists $x \in \mathbf{R}$ such that $x^2 = 2$.

   (a) Let $T = \{t \in \mathbf{R} \mid t^2 < 2\}$. Explain why T has a least upper bound and call it x.

   (b) You will prove that $x^2 = 2$ by showing that $x^2 \not< 2$ and $x^2 \not> 2$.

   (c) To get a contradiction first assume that $x^2 < 2$.

   (d) As an aside for later use show that for any $n \in \mathbf{N}$ $(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2x+1}{n}$.

   (e) Now, back to the main idea explain how $x^2 < 2$ implies $\frac{2-x^2}{2x+1} > 0$.

   (f) Choose $n_0 \in \mathbf{N}$ such that $\frac{1}{n_0} < \frac{2-x^2}{2x+1}$ (why can you find such an n_0).

   (g) Explain how this shows that $\frac{2x+1}{n_0} < 2 - x^2$.

   (h) Rewrite the previous inequality as $x^2 + \frac{2x+1}{n_0} < 2$.

   (i) Explain how shis shows that $(x + \frac{1}{n_0})^2 < 2$.

   (j) Thus $x + \frac{1}{n_0} \in T$ (why?). Why is this a contradiction.

(k) Now assume that $x^2 > 2$ and choose m_0 such that $\frac{1}{m_0} < \frac{x^2-2}{2x}$. Why can you do this?

(l) Show then that $\frac{2x}{m_0} < x^2 - 2$.

(m) Now, $(x - \frac{1}{m_0})^2 = x^2 - \frac{2x}{m_0} + \frac{1}{m_0} > x^2 - \frac{2x}{m_0} > 2$ (why is the last strict inequality true?).

(n) Thus $x - \frac{1}{m_0} < x = lub(T)$. Explain why this gives a contradiction.

4 Homework II on Analysis

Definition 4.1. *Given a sequence* $< a_n >$ *we say that it converges if* $\exists A \in \mathbf{R}$ *such that* $\forall \epsilon > 0, \exists N \in \mathbf{N}$ *such that* $\forall n > N$ *we have* $|A - a_n| < \epsilon$

1. Show that the sequence $< \frac{1}{n^p} >$ converges if $p > 1$. Find its limit.
2. Show that the sequence $< \frac{2n}{3n+7} >$ converges. Find its limit.
3. F& P # 10.
4. F& P # 6.
5. F& P # 7.
6. Using no negative words say what it would mean for a sequence $< a_n >$ not to converge. Then, show that $< 3^{2n-1} >$ does not converge.

   [Hint:

   (a) Take an arbitrary $M \in \mathbf{R}$.

   (b) Explain why $3^{2N-1} \geq M$ if and only if $(2N-1)log_{10}(3) > log_{10}(M)$.

   (c) Use this to get an express of the form $3^{2N-1} \geq M$ if and only if $N \geq ($ an epression involving M and logs).

   (d) Now choose N accordingly.]

5 Homework III on Analysis

1. Let $< a_n >$ be defined recursively as follows: $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n}$. Use induction to prove that $a_n = 3^{\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}}$.

2. Let $< a_n >$ be defined as above. Prove that $< a_n >$ is increasing and thus bounded below.

3. Let $< a_n >$ be defined as above. Prove that $< a_n >$ is bounded above by 3.

4. Concluded that $< a_n >$ converges.

5. Find $\lim_{n\to\infty} \frac{3n^2+n+4}{3+n+5n^2}$.