NAME:____
FINAL: Fundamentals of Mathematics (Math 320) Stevenson

General Instructions

1. **You are allowed your cheatsheat.**
2. **NO Calculators.**
3. **Please show all your work, unless explicitly instructed not to do so.**
4. **Please ask if you are not sure of anything on the exam.**
5. **You have 120 minutes.**

Question	Credit	Total Points
1		20
2		20
3		20
4		20
5		20
Total		100

1. For each of the following circle your answer.

   (a) Consider the congruence equation $10x + 14y = 232$. How many solutions exist? (Circle the correct answer.)

   none   infinitely many   2   5   7

   (b) Let $p_1, p_2, ..., p_r$ be primes. Then $p_1 | (p_1 p_2 ... p_r + 1)$.

   True   False

   (c) Let $S : P_2 \to P_3$ via $S(f(x)) = xf(x)$. Then S is:
   (circle **all** that are correct):

   Injective   Surjective   Neither

   (d) Let $< a_n = (-1)^n \frac{4}{2n+7} >$. Then $< a_n >$ is :
   (circle **all** that are correct):

   Bounded   Convergent   Neither

   (e) $(123)^6 \equiv 1 (\text{mod } 5)$.

   TRUE   FALSE

2. Let $S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x^2 - y^2 = 0\}$.

   (a) List all elements of the class $S[3]$.

   (b) Prove that $\sim$ is an equivalence relation on $\mathbf{R}$.

3. Statements: Let $U \subset \mathbf{R}$.

Definition 0.1. *A function* $f : U \to \mathbf{R}$ *is* **continuous on** U *if* $\forall x_0 \in U$ *we have* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $\forall x \in U$, *if* $|x - x_0| < \delta$ *then* $|f(x) - f(x_0)| < \epsilon$.

Definition 0.2. *A function* $f : U \to \mathbf{R}$ *is* **uniformly continuous on** U *if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $\forall (x_0, x) \in U \times U$ *if* $|x - x_0| < \delta$ *then* $|f(x) - f(x_0)| < \epsilon$.

(a) Without using negative words, state what it means to say that $g(x) = e^x$ is not uniformly continuous on $U = (0 < x < \infty)$.

(b) State the contrapositive to the following statement: If U is compact and $f : U \to \mathbf{R}$ is continuous then $f : U \to \mathbf{R}$ is uniformly continuous.

(c) In which of the two definitions above is the choice of δ allowed to depend on x_0? Why?

4. Let $M_2(\mathbf{R})$ be the set of 2×2 matrices with real entries. That is:

$$M_2(\mathbf{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbf{R} \right\}.$$

Recall that we can add matrices as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$$

Let $S_2(\mathbf{R})$ be the set of 2×2 symmetric matrices with real entries. That is:

$$S_2(\mathbf{R}) = \left\{ A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} | x, y, z, w \in \mathbf{R} \ \& \ y = z \right\}.$$

Define $T : M_2(\mathbf{R}) \to S_2(\mathbf{R})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

(a) Is $M_2(\mathbf{R}) \subset S_2(\mathbf{R})$? Prove your answer.

(b) Using the function T defined above, compute $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)$.

Still using $T : M_2(\mathbf{R}) \to S_2(\mathbf{R})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

(c) Prove that the codomain is correct. That is, compute $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and explain why it lies in $S_2(\mathbf{R})$.

(d) Is f surjective (onto)? Prove your answer.

(e) Is f injective. Prove your answer.

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5. We'll define a sequence of numbers as follows:
$a_1 = 1$, $a_2 = 1$, and for each natural number $n > 2$, let $a_n = 2a_{n-1} + 3a_{n-2}$.

(a) Find a_3 and a_4 and observe that $a_3 > 3$ and $a_4 > 3^2$.

(b) Use induction to prove that for all $n \geq 3$, $a_n > 3^{n-2}$.
(Hint: Define $S = \{n \in \mathbf{N} \mid a_n > 3^{n-2}\}$. Use part (a) to show that 3,4 are in S. Now assume that $\{4, ..., n\} \subset S$ and show that $n + 1 \in S$. Use back of page if necessary.)

(c) Part (b) also shows that $a_n > 0$ for all n. Use that and the recurrence relation to show that $< a_n >$ is increasing.

(d) Show that 1 is the greatest lower bound of $A = \{a_n \mid n \in \mathbf{N}\}$.

(e) Prove that the sequence $< b_n = \frac{1}{a_n} >$ converges.

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