Part 4 – Time Value of Money

One of the primary roles of financial analysis is to determine the monetary value of an asset. In part, this value is determined by the income generated over the lifetime of the asset. This can make it difficult to compare the values of different assets since the monies might be paid at different times. Let's start with a simple case. Would you rather have an asset that paid you $1,000 today, or one that paid you $1,000 a year from now? It turns out that money paid today is better than money paid in the future (we will see why in a moment). This idea is called the **time value of money**. The time value of money is at the center of a wide variety of financial calculations, particularly those involving value. What if you had the choice of $1,000 today or *$1,100* a year from now? The second option pays you more (which is good) but it pays you in the future (which is bad). So, on net, is the second better or worse? In this section we will see how companies and investors make that comparison.

Discounted Cash Flow Analysis

Discounted cash flow analysis refers to making financial calculations and decisions by looking at the cash flow from an activity, while treating money in the future as being less valuable than money paid now.   In essence, discounted cash flow analysis applies the principle of the time value of money to financial problems. In part 5 we will see how discounted cash flow analysis can be used to value a variety of different kinds of assets. In this section, we will concentrate on the basic math behind the time value of money and apply it to situations involving borrowing and lending.

The math behind the time value of money and discounted cash flow analysis shows up in a number of different places. For example, each of these questions involves monetary payments made at different points in time:

- We put away $100 per month in a savings plan. How much will we have in 10 years?

- We are planning to put a down payment on a house in 5 years. If we save a regular amount every month, how much will we need to save each month?

- If we have a certain amount of money in retirement savings, and we need to live off it for the next 20 years, how much can we withdraw each month?

- We can make mortgage payments of $900 per month. How much can we borrow?

- We need to borrow $10,000 now and repay it over the next three years. How much must we pay per month?

All of these are discounted cash flow problems and can be solved using the techniques presented in this section.

When solving a discounted cash flow problem, it is best if you take in steps. The first step is to determine when all the payments are made and then list the payments. One of the best ways to represent the payments is on a time line.

Constructing the Time Line

A time line is a graphical representation of when payments are made. Say that you get a loan of $25,000 that requires you to make three equal payments of $10,000 at the end of the next three years. We could write out the payments as:

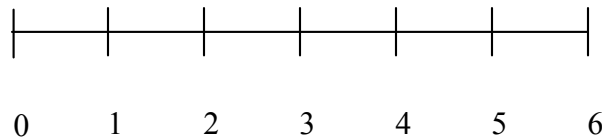| Time	Payment
Now	+$25,000
End of this year	-$10,000
End of next year	-$10,000
End of year after	-$10,000

The first column shows when the payment is made, the second column shows the amount of the payment (a "+" means the cash is "flowing in" while a "-" means the cash is "flowing out").

The time line shows the timing of these payments on a line (we don't call it a time line for nothing!) The convention is to number the start of the first year as 0, the end of the first year (the start of the second year) as 1, the end of the second year (the start of the third year) as 2 and so on. A time line showing the current year and the following 5 years looks like this.

```
|    |    |    |    |    |    |
0    1    2    3    4    5    6
```
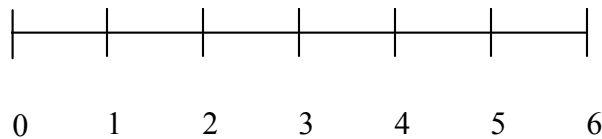
We can write the payments on the time line to indicate when they are paid. For example, in our three-year loan,

+25,000 -10,000 -10,000 -10,000

```
|    |    |    |    |    |    |
0    1    2    3    4    5    6
```

For this simple loan there really isn't any advantage to drawing the time line. However, it will be very useful when looking at more complicated assets.

Time lines are quite versatile. If the payments happen every month, instead of every year, we let the numbers represent months. 0 is now the start of the first month, 1 is the end of the first month, 2 is the end of the second month and so on.

Most of this section will have to do with moving money across time: How much will an investment now be worth in the future, or how much is a promise of money paid in the future worth now? We can indicate the shift of money across time using the time line.

For example, suppose we are working as a financial advisor. Someone has inherited $10,000 that the want to save for their retirement in 5 years. If the interest rate is 5%, how much money will this give them when they retire? This is a "future value problem" (which we will learn how to solve shortly). The time line is given by:
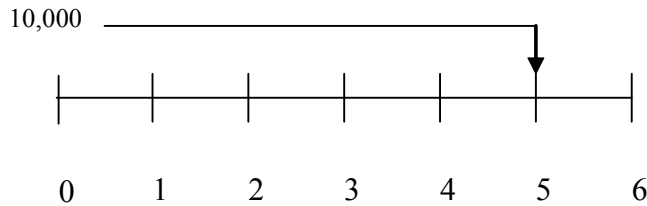
Other times we are interested in knowing what money paid in the future will be worth today. For example, you are getting a payment of $10,000 in 4 years, but you want to borrow against that money now. How much can you borrow? In other words, what is the equivalent *now* to having $10,000 in the future? This is called finding the present value of a payment.
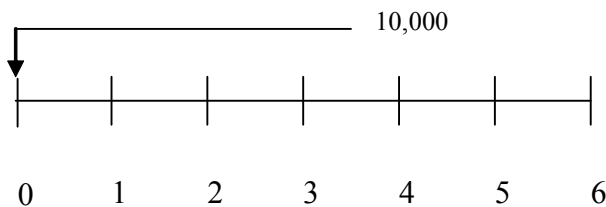
As we will see, these really aren't different problems. It turns out that there is a simple formula that connects money paid at different times.

Present and Future Values of a Single Amount

Finding the Future Value of a Present Amount

Say that you put $1,000 into the bank today. How much will you have after a year? After two years? This kind of problem is called a **future value** problem. We want to know the value in the future of an amount today. It is also called a **compounding** problem, because, as we will see, the values compound, or multiply, over time. On a time line, our problem looks like this,

We are trying to determine the value of $1,000 moved one year into the future. If you know the interest rate, it is simple to find the answer. For example, if you have an annual interest rate of 6%, then you will have $1,060 one year from now ($1,060 = $1,000×1.06). Another way of

saying that is, the future value of $1,000 one year from now at an interest rate of 6% is $1,060. If you left the money in the bank for two years, you would have $1,060 after the first year, and $1,123.60 after two years ($1,123.60 = $1,000×1.06×1.06). In other words, the future value of $1,000 two years from now at an interest rate of 6% is $1,123.60.

We would like to come up with a *formula* that combines the amount we invest with the interest rate to tell us the future value. In our bank example, we get the future value ($1,123.60) by multiplying the present value ($1,000) by the gross interest rate twice. As a formula, we express it this way,

$$\$1,123.60 = \$1,000 \times (1.06)^2$$

More generally, if we let *FV* stand for future value, and *PV* for present value (the amount today), and *k* for the interest rate, we can express the relationship this way,

$$FV = PV \times (1+k)^2$$

Now, let's make the formula apply no matter how many years we leave the money in the bank. We replace the "2" by the letter *n*, indicating the number of years, to get the formula,

$$FV_n = PV \times (1+k)^n$$

where,

   FV_n = future value at the end of period n
   PV = present value
   k = annual rate of interest
   n = number of periods

This is our formula for the future value of a current amount *n* years in the future, at interest rate *k*.

   Example: How much is $10,000 worth 6 years from now if the interest rate is 5%?

   PV=$10,000, k =0.05, n = 6. Using our formula, $FV_6 = \$10,000(1.05)^6 = \$13,400.96$.

We can see from the formula why a dollar today is better than a dollar in the future. A dollar today can be invested, and by investing, you will end up with more than a dollar in the future.

Finding the Present Value of a Future Amount

Let's turn the last example around. Say that we just happen to need $13,400.96 in 6 years. If the interest rate is 5%, how much would we need to put in the bank now? Well, we know the answer to this question from the previous example; we need $10,000. $10,000 in the present is the same as 13,400.96 six years in the future (at a 5% interest rate). It doesn't matter if we are going from the present to the future, or from the future to the present, the relationship between the payments is the same. We can show this equivalence on a timeline.
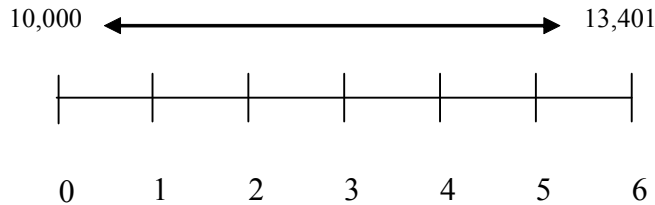
The **present value** of a future payment is the amount that the payment is worth today. Therefore, the present value of $13,401.96 paid 6 years from now at an interest rate of 5% is $10,000.

Whenever we need to find the present value of a future amount, we can use the future value formula, just rearranged. Take our future value formula,

$$FV_n = PV \times (1+k)^n$$

and rearrange to isolate the present value term,

$$PV = FV_n / (1+k)^n$$

Of course, these are not two different formulas, they are just two ways of showing the relationship between money at two different points in time. Anytime we know three of the variables we can find the fourth.   Whenever we are borrowing or lending money, the four variables: time, interest, present and future values will always be there, and this formula is the math that ties them all together.

We will run through some examples using our present value formula.

> **Example**. You buy a refrigerator for $800 but you don't have to make the payment until next year (that is, one year later). The opportunity cost of money is 3%. Opportunity cost represents what you could earn with the money – in this case, the return on your best investment opportunity. What price are you paying for the refrigerator in present dollars?
>
> In other words, the problem is asking for the present value of $800 paid one year from now, at an interest rate of 3%. We know that the present value will be less than $800, but how much less? In our formula, $n = 1$, $k = 0.03$ and $FV = \$800$. The present value is given by $PV = \$800/(1.03)$. That is, for a person who can invest money at 3%, $800 one year from now is the equivalent of $776.70 today. Because you could defer payment for a year, the refrigerator actually costs you less than $800. By waiting one year to make the payment, you only need $776.70 now to buy the refrigerator.

Our present value formula tells us couple of things about money. In our equation, as n gets larger, the present value gets smaller. In other words, the farther into the future that money is paid, the less it is worth.

> **Example (continued).** You can defer payment on the refrigerator for an additional year. What is the equivalent current price of the refrigerator?
>
> We redo the calculation with $n = 2$ and we get: $PV = \$800/(1.03)^2 = \$754.08$, which is smaller than the previous value. The farther into the future we can push the payment, the

lower the cost of the refrigerator. The reason for that is that we can invest the money for a longer period and earn more interest before we have to make the payment.

Our present value formula also tells us that as k increases, the present value decreases. In other words, the higher the interest rate, the lower the present value of money paid in the future.

Example (continued). If interest rates were 20% and you can pay for the refrigerator in two years, what is the equivalent current price?

Using the same formula, with $k = 0.20$, we get PV = $\$800/(1.20)^2 = \$555.56$, so the refrigerator costs us less. The reason it costs less is that there is now a greater value to deferring our payment. Before, we could only earn 3% on our investment, so the value of having that investment opportunity wasn't that much. Now we can earn 20%. So allowing us to defer the refrigerator payment and earn 20% for two years is a substantial benefit and significantly reduces the cost of the refrigerator.

Determining the Interest Rate and the Opportunity Cost of Money

Interest rates show up in present value and future value problems because they tell us the cost or benefit of moving money across time. Sometimes we are in situations where we are told about a present payment and a future payment but not explicitly about interest rates. However, we can use our present value formula to figure out what the interest rate is.

Example: You can buy a car for $30,000 now, or pay for it one year from now $35,000. What interest rate are you being offered?

No surprise, it's the same formula as before, except this time we know the present value and the future value and we are asked to find the interest rate. The math behind this kind of problem can be a bit harder to solve, but in this example it is straightforward. PV = $30,000, FV = $35,000, and n = 1, so that,

$\$30,000 = \$35,000/(1+k)$

$k = 16.67\%$

The use of present value allows us to compare prices (or any financial payments) at different points in time. In the example above, it would only make sense to wait until next year to make the payment if the value to us of not paying now was greater than 16.67% (for example, if we could use the money for an investment opportunity with a return of 25%)

The interest rate you use can make a big difference in the calculation. Large companies go to great effort to determine the appropriate rate and we will see how they do it in a later section. However, if you are in doubt about what interest rate to you, remember that the key principle is the economist's idea of opportunity cost. Ask yourself what opportunity you would give up to take an action, and the interest rate for that opportunity is what you would use.

Present Value of Complicated Payment Streams

One of the benefits of the present value approach is that it can be used with complicated payment streams.   The present value of a stream of payments is just equal to the sum of the present values of the individual payments.

> **Example:** Say that you are buying a refrigerator, and make a $400 payment one year from now and a $400 payment two years from now. What is the present value of your total payment given an interest rate of 3%?
>
> The answer is the sum of the present values of the individual payments. The present value of the first $400 is $388, the present value of the second $400 is $377, and so the present value of both payments is $765.

The fact that we can add up the present value of payments is critical. Most financial situations do not involve a single future payment, rather they usually consist of a number of different payments at different times. However, we can still use our simple present value formula to determine their total value. We will do this shortly; however, first we must take care of a complication.

Compounding

When you make a loan you expect to earn interest on the principle (the amount of the loan). But you can also earn interest on the interest you are paid, a process called **compounding**. The compounding of interest over a number of years can dramatically increase your earnings. Imagine that you deposit $1,000 in the bank at 10% interest compounded annually (compounded annually means that interest is paid once at the end of the year). At the end of the first year you get back $100 of interest along with your principal of $1,000. You reinvest your principal, but keep out the interest so that it doesn't earn any additional interest. At the end of 20 years you would have a total of $2,000 of interest payments along with your principal of $1,000, for a grand total of $3,000. If you had reinvested both your interest and principal, the value of your investment after 20 years would be found using our future value formula, $1,000(1.1)^{20} = $6,727. The $3,727 difference is entirely due to interest earning interest, the process of compounding.

So far, we have been assuming annual compounding. However, interest is sometimes paid out over the entire year. Just like with annual compounding, this is good for the investor, since interest earned earlier in the year will earn interest of its own for the remainder of the year.

Since it is very common for interest to be compounded more frequently than once a year, we need to include it into our calculations. When quoting interest rates it is typical to speak in terms of annual rates and their period of compounding. For example, one might refer to 12% interest compounded monthly. The "12%" is called the nominal rate. Nominal just means "named" and is given in annual terms, since people are more comfortable in dealing with annual interest rates. After the interest rate comes the compounding period – in this case monthly – which tells us how often the interest is paid – in this case once a month. The amount of money you earn will depend on both the nominal rate and the compounding period.

We will go through the most common compounding periods and see how much interest you would earn on $1,000.

● 12% interest compounded *annually*.

Interest is paid once, at the end of the year. The total payment equals $1000×(1.12)=$1,120. We can show when the interest payments are made on a timeline that is measured in months.

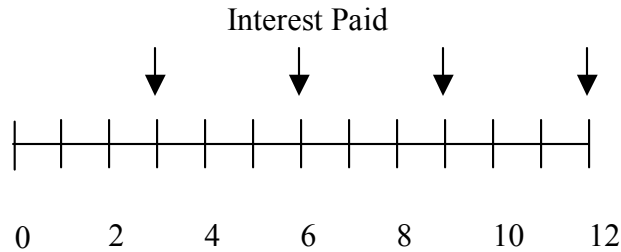Interest Paid

0   2   4   6   8   10   12
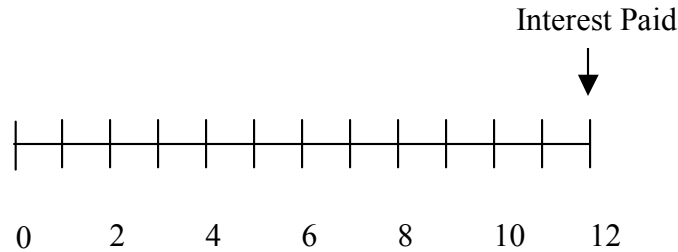
● 12% interest compounded *semiannually*.

"Semiannually" means twice per year so we divide the 12% into two equal parts. The first 6% is paid after 6 months and the second 6% at the end of the year. After 6 months we have $1,000(1.06)=$1,060, which then earns interest over the remainder of the year. The total amount at the end of the year equals $1,000(1.06)(1.06) or $1,123.60. This is more than $1,120 since the $60 of interest paid after 6 months earns interest over the rest of the year.

Interest Paid

0   2   4   6   8   10   12
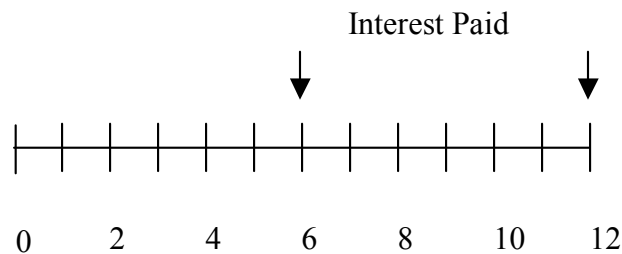
● 12% interest compounded *quarterly*.

Interest is now paid in 4 installments. We split the 12% into 4 equal parts of 3% each. The total amount is now $1,000(1.03)^4 = $1,125.51

Interest Paid

0   2   4   6   8   10   12

● 12% interest compounded *monthly*.

Interest is now paid in 12 installments. We split the 12% into 12 equal parts of 1% each. The total amount is now $1,000(1.01)^{12} = $1,126.83.

Interest Paid

0   2   4   6   8   10   12
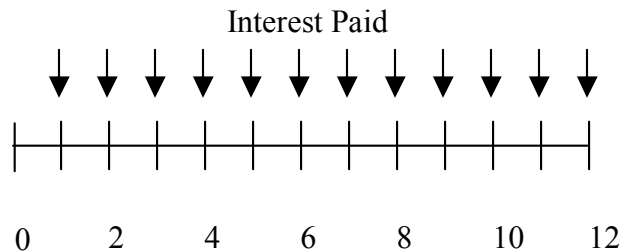
● 12% interest compounded *daily*.

For financial purposes, there are various conventions on how many days there are in a year. We will assume a standard 365-day year. The daily interest rate is 12/365 and total amount is give by $1,000(1+0.12/365)^{365} = $1,127.47. As the compounding period continues to get smaller, the total amount increases, although the dollar amount of the increase is not large.

● 12% interested compound *continuously*.

We can imagine the compounding period getting smaller and smaller until interest is effectively paid continuously. The formula for the future value of money compounded continuously is a special case, and is given by $FV_n = PV(e^{kn})$, where *e* is a special mathematical constant. In our example, k = 0.12, n = 1 and PV = $1,000 so that the future value is given by $1,127.50.

Present and future value calculations with compounding periods less than a year

So far, we have looked at the effect of annual compounding across many years, and the effect of more frequent compounding within a year. What do we do when we have more frequent compounding and we are looking across many years? For example, we are depositing $100 in a bank at 6% interest compounded monthly and we will leave it there for three years. How much will we have at the end?

There are basically three ways to do this. The first is convert all numbers to a monthly frequency and use our future value formula. For our example, 6% interest compounded monthly translates to 0.5% interest per month; therefore, k = 0.005. Time periods are now measured by months, so 3 years translates to 36 months; therefore, n = 36. Using our formula, the future value equals $100(1.005)^{36} = $119.67. For most of the rest of the course, this is how we will solve this kind of problem, so we should do another example.

> **Example:** We want to find the present value of $20,000 paid 12 years from now using a discount rate of 10% compounded quarterly.
>
> The quarterly interest rate is 2.5% or k = 0.025. The number of periods is 12×4 or n = 48. By our formula, we have $20,000/(1.025)^{48} = $6,113.42.

A second way of doing this type of problem is to use financial calculators that can make the adjustment for us – we will discuss this approach in the section on calculators. The third way of doing this is to calculate an **effective annual rate** of interest.

Effective Annual Rate of Interest

We know that the same nominal rate can generate different returns depending on the compounding period. If we are comparing different nominal rates with different compounding periods, it gets even more complicated. We need a way of coming up with one number that incorporates both the nominal rate and the compounding period.

The effective annual rate of interest (EAR) does just that. It calculates the rate of interest *compounded annually* that gives you the same amount of interest as the listed nominal rate.   For example, we know from a previous problem that $100 invested at 12% compounded monthly gives you 112.68. Therefore, 12% compounded monthly is just the same as 12.68% compounded annually, and so 12.68% is the EAR.

 The formula for the EAR is,

$$EAR = (1+k_{nom}/m)^m - 1$$

where k_{nom} is the nominal interest rate and *m* is the number of compounding periods. The big advantage of EAR is it makes interest rates easier to compare, which is why banks are often required to list these rates in addition to nominal rates.

> **Example:** Two banks offer different interest rates. One offers 10% compounded annually. The other offers 9.75% compounded monthly. Which is better?
>
> The EAR for 9.75% compounded monthly is 10.20%, so that rate is better.

We can also use EARs to make multiyear calculations since once we have an EAR we can use the standard annual compounding formulas.

> **Example:** We are depositing $500 in a bank at 6% interest compounded monthly and we will leave it there for three years. How much will we have at the end?

The first step is to convert the 6% nominal interest to an EAR. 6% compounded monthly has an EAR of 6.17%. The second step is to use our future value formula with annual values: k = 0.0617 and n = 3. The future value = $500(1.0617)^3 = \$598.34$.

All three approaches to these multi-period compounding problems will give the same answer. You should generally choose the method that is most convenient. (But for this course, we will almost always use the first method).

4) Using a Financial Calculator and Spreadsheets

Financial calculators are special calculators that have financial functions, such as present value, built in. We didn't need to use special functions in the previous sections, since simple present value problems can be solved by hand (or at least by simple calculators); however, we will need these functions later when we examine annuities, so we will take this opportunity to introduce the calculators.

At this point you should know how to use the basic functions on your calculator including Clear and Recall. [Note: There will be a special calculator handout on the class webpage].

First, there are two settings that need to be set on the calculator.   The calculator needs to know if the payments are made at the end of the period or at the start (this will matter when we calculate the value of annuities). Generally, we want payments made at the end of the period. This should be the default. The HP 10B will say "begin" on the screen if it is in the begin mode but nothing if it is in the end mode. Use the [BEG/END] key to switch between models.

The calculator also needs to know the number of periods per year if we enter all variables on a yearly basis. For example, if we were compounding monthly we could enter all the variables (interest rate, years, etc) as annual numbers and then tell the calculator that there are 12 periods per year. However, like we did earlier, we will convert all variables to the compounding period. Therefore, we need to set [P/YR] to 1. To do this on the HP 10B, hit [1], [orange],[P/YR]. To check if you did it right, clear the screen using the [C ALL] button. The screen should say 1 P_yr.

Now we are ready to do some calculations. There are 5 keys that store the values we need for our formula. The keys are:

> N   : Number of periods
> I/YR   : Interest rate (per year) (note: *7% is indicated by 7, not 0.07!*)
> PV   : Present Value
> PMT   : Payment (we will use this for annuities, later)
> FV   : Future Value

To enter a number for each of these variables, press the number and then press the key for the variable you want. For example, to set the number of periods equal to 6, press [6] then [N]. The calculator will remember that N = 6 until you enter a different value for N, or turn the calculator off.

If we press one of the variable keys without entering a number first, the calculator will calculate the value of that variable, given the values of the other variables. The way we will use the calculator to solve problems is to store four of the values, and let the calculator solve for the fifth.

Let's do a future value problem.

Example: If we invest $10,000 now (and make no further payments), how much will we have after 10 years, if interest is 8% compounded annually.

We could solve this directly using the formula to get $(10,000)(1.08)^{10} = 21,589.25$. Using the calculator, we enter the following values.

        N   : 10
        I/YR   : 8
        PV   : -10,000 (This is a negative number since it is a payment *out*)
        PMT   : 0   (This will equal zero until we get to annuities)
        FV   :

!! Important note!! Some calculators want payments out entered as positive numbers. Read your calculator manual!

We know the amount today ($10,000) but we are trying to find the amount in the future. So, after entering the other numbers, we press the future value (FV) key and we get:

        FV = 21,589.25

This is a positive number, because it is a payment to us.

If you are not getting this number then check to make sure you entered the correct numbers and that your settings are correct. (For example, if you get 10,687 then your value of P/YR is 12, not 1)

When doing problems, you always want to ask yourself 'is the answer reasonable?' It is easy to press the wrong buttons, so check your work. For example, in the last problem, if you came up with the answer $5,000, you know it would be wrong (it has to be a number larger than $10,000).

Here are some more problems

Example (continued): How long would we have to wait until we get $30,000?

In this problem, we know the value in the future, but we don't know the time. We enter the four variables we know, and then press N.

        N   : ?
        I/YR   : 8
        PV   : -10,000
        PMT   : 0
        FV   : 30,000

The answer is, N=14.27, or just over 14 years and 3 months.

Example (continued): If we wanted to get to $30,000 in 10 years, what interest rate would we need?

```
N      : 10
I/YR   : ?
PV     : -10,000
PMT    : 0
FV     : 30,000
```

The answer is I/YR = 11.61, or 11.61%.

Example: If we are paid $100,000 5 years from now, what is the present value, if we discount at 4% (compounded annually)?

```
N      : 5
I/YR   : 4
PV     : ?
PMT    : 0
FV     : 100,000
```

The answer is PV = 82,192.71. (The calculator will show this as –82,192.71)

Example (continued): What is the present value if the 4% is compounded monthly? To find the answer we first translate the annual values into monthly values. 4% annual nominal rate implies a monthly rate of 0.3333 (4/12). (Does it matter how you round the interest rate? It can. Try it and see). Five years equals 60 months, so we enter the following values:

```
N      : 60
I/YR   : 0.3333
PV     : ?
PMT    : 0
FV     : 100,000
```

The answer is PV=81,901.94

Annuities and Perpetuities

What is an Annuity?

Most problems in financial analysis have payments made in more than one period. For example, if you have a 30-year mortgage, you are required to make 12 payments per year for 30 years, for a total of 360 payments. If we wanted to find the present value of the payments you could find the present value of each of the 360 payments individually; however, that would take a long time. Fortunately, there is an easier way to do this.

We can take advantage of the fact that each of the payments of the mortgage is the same.

This kind of regular payments is called an annuity (it gets its name from the word "annual" meaning yearly – but it applies to any regular payment). We can take advantage of the fact that it is an annuity to simplify the present value calculation.

There are basically two types of annuities, ordinary annuities, which start the payments at the end of the period, and annuities due, which start the payments at the beginning of the period (we will cover those later in this section). An ordinary annuity that paid $1,000 for three years would look like this on a timeline.
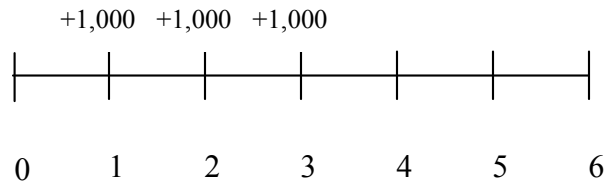
The Future Value of an Annuity

Say that you deposit $1,000 in a savings account at the end of each of the next three years (that is, at the end of year 1, year 2 and year 3). How much will you have at the end of year 3, if the interest rate is 5% (compounded annually)?

On a time line, the problem looks like this,
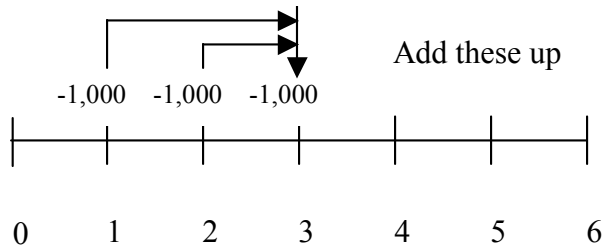
This is an ordinary annuity as we have regular payments made at the end of successive years. It is a future value problem since we need to find the total value of the payments at a point in the future. One way to do this is to calculate the future value of each of the payments individually, and then add them together.

$$FV = \$1,000 + \$1,000(1.05) + \$1,000(1.05)^2$$

The first number on the right hand side is the payment made in year three. The second number is the payment made in year two, saved for one year. The third number is the payment made in year one, saved for two years.

We can write the problem more generally. Let FVA stand for the future value of an annuity, PMT stand for the annual payment of the annuity, and k, as always, stand for the interest rate. The future value of a three-year annuity is given by,

$$(FVA) = PMT + PMT(1+k) + PMT(1+k)^2$$

If there were more than three years, we would just add additional terms to the right hand side. Using this formula, anyone could calculate the future value of the annuity if you told them three things: The regular payment (PMT), the interest rate (k) and the number of payments made (n).

This is what a financial calculator or a spreadsheet does. We give it the information about PMT, k and n, tell it to find the future value, and it goes to work.

> **Example.** If you are saving $100,000 each year for the next 10 years at an interest rate of 7% compounded annually, how much will you have at the end of the 10 years?
>
> Since this is a future value problem, we put 0 into PV. The payment is –100,000 (it is negative since it is a flow out). N = 10, and I = 7. Our list of variables is.
>
> N   : 10
> I/YR   : 7
> PV   : 0
> PMT   : -100,000
> FV   : ?
>
> We press the future value key and we get 1,381,644.80, which is our answer. Saving $100,000 each year for 10 years will give us $1,381,644.80.

We will run through some additional annuity problems. In each problem, the important thing to determine is, which variables do you know and what are you trying to solve for?

> **Example (continued):** If you wanted to have $1,500,000 then how much would you have to save?
>
> N   : 10
> I/YR   : 7
> PV   : 0
> PMT   : ?
> FV   : 1,500,000
>
> The answer is PMT = -108,566.25. You would have to save $108,566.25 each year.

> **Example:** How long will it take to get $20,000 if you save $300/month at an 8% return compounded monthly?
>
> In this problem we want to know the length of time we need to save. The one trick to the problem is that the interest rate is an annual rate but we need to convert everything to a monthly basis. An 8% annual rate = 0.67% monthly rate.
>
> N   : ?
> I/YR   : 0.67
> PV   : 0

```
PMT    : -$300
FV     : $20,000
```

Our answer is N = 55.3. Since this is in months, we convert back into years by dividing by 12. Our final answer is 4.6 years.

Example (continued): Say that 4.6 years is a bit too long.   If we want to accumulate enough money in 4 years then we need to either save more or earn a higher rate of return.

What return would you need to get $20,000 in 4 years saving $300/month?

We input:

```
N      : 48
I/YR   :  ?
PV     : 0
PMT    : -$300
FV     : $20,000
```

Our answer is 1.3351, which is a monthly interest rate. To get an annual nominal interest rate we multiply by 12 to get 16% per year compounded monthly.

Example (continued): How much do you need to save per month to get $20,000 in 4 years at an 8% return?

We input:

```
N      : 48
I/YR   :  0.67
PV     : 0
PMT    : ?
FV     : 20,000
```
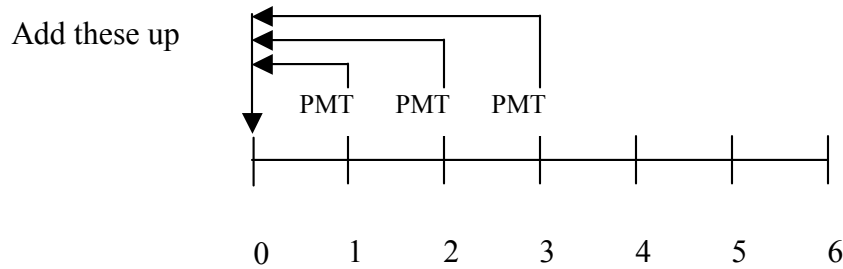
The payment equals -354.63 meaning that we must save $354.63 per month.

The Present Value of an Annuity

Now that we are experts at present value math, finding the present value of an annuity should be no problem. Say that you are given a fixed payment each year for the next three years. What is the present value? First, the timeline,

The formula for the present value of an annuity (PVA) is the sum of the present values of the individual payments.

$$(PVA) = PMT/(1+k) + PMT/(1+k)^2 + PMT/(1+k)^3$$

The first term on the right hand side is the present value of the payment one year from now. The second term is the present value of the payment two years from now, and so on. Again, we can do the calculation by hand, but for larger problems it is easier to provide the calculator or spreadsheet with the information and have it do the math.

Present value problems show up when discounting cash flows or when determining the value of loans.

Example: After a lawsuit, you have received an award that pays you $20,000 semi-annually for 16 years. A bank is willing to buy this stream of payments from you, discounting the payment at an interest rate of 12%. How much will you get?

The first thing is to recognize that this is a present value problem. The company is getting money from the bank now, in exchange for giving up the future payments. We set FV = 0 and solve for PV. The second key to the problem is that we need to convert everything to a semi-annual basis.

```
N       : 32              (16 years times 2 periods per year)
I/YR    : 6
PV      : ?
PMT     : -20,000         (Payment is negative since we are giving it up)
FV      : 0
```

The answer is that the present value equals $281,681. That is the amount that the bank will give you.

Example (continued): If you wanted to get 350,000 from the payments, what interest rate are you asking for?

It is still a present value problem, so FV = 0. The difference is that this time we know what the present value is and we need to solve the interest rate. We input:

```
N      : 32
I/YR   :  ?
PV     : 350,000
PMT    : -20,000
FV     : 0
```

The answer is a semi-annual interest rate of 4.17, which makes for a 8.34% *annual* nominal interest rate.

Present value problems are at the heart of every loan. The present value of your loan payments is calculated to equal the amount of money you borrow.

Example: You borrow $20,000, which you will pay off 3 years at 18% interest compounded monthly. What is your loan payment?

This is a present value problem since you are receiving the payment today. You need to convert the variables to their monthly value and find the amount of the payment. We input:

```
N      : 36
I/YR   :  1.5
PV     : 20,000
PMT    : ?
FV     : 0
```

The answer is, PMT = -723.05, which means that each monthly payment will be $723.05.

Example: You can get a 4-year loan at 12% compounded monthly.
If you can afford to pay $400/month, how much can you afford to borrow?

Again, this is a present value problem. This time you know the payments and need to find the present value. You input:

```
N      : 48
I/YR   :  1
PV     : ?
PMT    : -400              (Payments are negative since they are an outflow)
FV     : 0
```

The answer is 15,189.58, so you can borrow $15,189.58.

An Annuity Due

The difference between an annuity and an annuity due is that the payments of the annuity due start right away (the start of the first month) rather than at the end of the first month (start of the second month).   Our three-year ordinary annuity looked like this,

```
         +1,000  +1,000  +1,000

   ├────┼────┼────┼────┼────┼────┤

   0     1     2     3     4     5     6
```
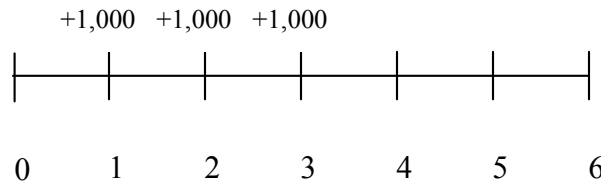
An annuity due that paid $1,000 for three years would look like this,

```
 +1,000  +1,000  +1,000

   ├────┼────┼────┼────┼────┼────┤

   0     1     2     3     4     5     6
```

An annuity due has a higher present value than an ordinary annuity since the payments are made earlier. In fact, each payment is paid one period earlier, so that each payment is one period's worth of interest more valuable. Because of this, the present value of an annuity due is just equal to the present value of the equivalent ordinary annuity multiplied by $(1+k)$.

It works the same way for the future value of an annuity due. When calculating the value at the final year, each payment of the annuity due has one extra period of compounding. This means that each payment is worth more, by the amount of the interest rate. So, to find the future value of an annuity due, you multiple the value of the ordinary annuity by $(1+k)$.

> **Example:** You are saving $1000 per month for the next three years at an annual interest rate of 6% compounded monthly. Each deposit is made at the start of the month. How much will you have at the end of the three years.
>
> Our strategy will be to calculate the value of an ordinary annuity first, and then make the adjustment for an annuity due. We input:
>
> | | |
> |---|---|
> | N | : 36 |
> | I/YR | : 0.5 |
> | PV | : 0 |
> | PMT | : -1000 (Payments are negative since they are an outflow) |
> | FV | : ? |
>
> The future value is $39,336. We multiply by (1.005) to get the annuity due value of $39,533.

Some financial calculators will make the annuity due adjustment for us. If we set the BEG/END key to BEGIN then the calculator knows that we want an annuity due. We input the numbers as always and it will multiply by the interest rate for us.

Perpetuities

Perpetuities are financial payments that are similar to annuities.   They have a regular stream of payments, but the difference is that the payments go on forever. It might seem like an infinite stream of payments would have an infinite value, but this is not the case. Since payments are worth less the farther in the future they are, the distant payments of the perpetuity become worth less and less until they are almost worth nothing. This puts a limit on the value of a perpetuity.

Perpetuities show up in a variety of different contexts. A special kind of bond issued in Britain, called a Consol, offers payments forever. Preferred stock offers dividend payments as long as the company survives, which as we will see, can be treated as infinite. Indeed, we shall see later that an entire company can be valued as a perpetuity.

The present value of a perpetuity (PVp) is given by the formula

$$PVp = PMT/k,$$

where k is the interest rate (*in decimal form!*).

> **Example**: What is the value of a perpetuity that pays $20,000 per year at a 10% interest rate?
>
> Using our formula, we get $20,000/0.1 = $200,000.

Because of discounting, most of the value of a perpetuity is determined by the payments in the earlier years. For example, in the last example, the perpetuity was worth $200,000. However, a 40-year annuity paying $20,000 at 10% interest is worth $195,581, which is almost as much. A 60-year annuity is worth $199,343. The payments of the perpetuity after 60 years add very little value. For this reason, we can think of the cash flow from a company as a perpetuity even though the company literally doesn't last forever. The value of the cash flows from the distant future is just too small to matter.

Amortization

We can use our knowledge of the time value of money to look at loan amortization. A standard loan has the borrower make payments of equal amount every period. Part of each payment goes to interest and part goes to paying down the principal. The reduction in principal over time is called amortization. How much of the payment goes to principal? This is an important question, particularly for mortgages, since interest payments on mortgages are tax deductible, while payments going to principal increase equity in the home.

It turns out that, early on, the borrower pays down little of the principle but gets a large interest deduction. Towards the end of the loan, the interest deduction is small, but the principle is being paid down rapidly. Let's see why this is.

Example: Say that you want a 4-year loan at a 12% interest rate compounded monthly. You can afford loan payments of $600 per month. This implies a loan balance of $22,784.38 (try it).

Let's look at what happens in the first month of the loan. The loan of $22,784.38 charges 1% interest each month, so the monthly interest cost is $227.84. If you paid *only that amount*, you would not have paid anything towards the principal, and so your ending balance would just equal your starting balance. This is shown on line (1) below.

	Balance	Interest Cost	Payment	Principal	Ending Balance
(1)	$22,784.38	$227.84	$227.84	$0	$22,784.38
(1')	$22,784.38	$227.84	$600	$372.16	$22,412.22

To pay down the loan, you need to pay more than the interest cost. Line (1') shows what happens if you made the full payment of $600. With this, you can pay the interest cost with the remainder going to reduce the principal. The ending balance is smaller than the starting balance – you are paying down the loan! The value of $600 is the value needed so you pay off the loan in exactly four years. Notice that the interest cost will be lower next month since the starting balance will be smaller. As an exercise, determine how much principal is paid next month assuming a payment of $600.

We can also look at the last month of the loan.

Balance	Interest Cost	Payment	Principal	Ending Balance
$594.06	$5.94	$600	$594.06	$0

Because the balance is so small, there is very little interest cost and almost all of the payment goes to principal.

Uneven Payments and Embedded Annuities.

Annuities are simple to calculate because all of the payments are the same. But what should you do if the payments are not the same? Generally, you have to calculate each of the payments individually. Financial calculators can do this too, but they require you to enter each of the payments individually, which can be tedious. A better way to approach this kind of problem is to use a spreadsheet since this will keep a permanent record of each of the numbers.

In some cases, you can look for embedded annuities in the problem, to shorten the calculation. An embedded annuity is a sequence of regular payments that make up part of the total payments. We can calculated the value of the annuity separately, and then add that part in to the total.

> **Example**: What is the present value of the following stream of payments assuming a 5% rate of interest compounded annually?

Year	Payment
1	-5,000
2	200
3	300
4…	
20	500

This pattern of cash flow is not uncommon- you might find it at a new project or company. First, there is an initial outlay of capital. As the business grows over time, cash starts coming in. At some point, the company reaches a steady rate of cash flow. Since there are 20 years of payments, it might seem that we have to do 20 different calculations to calculate the total present value. However, we can shorten this by taking out the embedded annuity that runs from year 4 to year 20.

First, we determine the present value of the first three cash flows.

> Present Value

> Year 1: $-5000/(1.05)$ = -4761.90

> Year 2: $+200/(1.05)^2$ = 181.41

> Year 3: $+300/(1.05)^3$ = 259.15

> = $-4{,}321.34$

Next we calculate the present value of an annuity that pays 500 for 17 years (Years 4 to 20) at a 5% interest rate. We input:

> N = 17
> I/Y = 5
> PV = ?
> PMT=500
> FV = 0

The present value is 5,637.03. This gives us the value of the annuity in *year 3*, the period before the payments start. We still need to take the present value (year 0) of this amount. To do this, we divide by $(1.05)^3$ to get \$4,869.48

We add this amount to the amount from the first three years and we get a total of \$548.14.

Compound Problems

In many situations, you will have more than one cash flow. For example, you might be saving using several different investments, or you might initially be building up your savings for your retirement, which you will then spend down. In principle, compound problems are no more difficult to do, but they do require extra care to make sure that all of the cash flows are correctly accounted for. Each time you are faced with one of these problems you should go through four steps.

1) Construct a time line of all the payments
2) See which payments are single amounts and which are annuities.
3) Determine if you are finding present values, future values, or the interest rate.
4) Determine which problems can be solved in isolation and solve them first.

Example: Your family is saving for a house. Your family income is $60,000 per year and you can afford to spend 25% of your income on mortgage payments. You want to buy a house costing $220,000 three years from now. You expect to be able to get a 30-year fixed rate mortgage at 9%.   You have $20,000 which earns 6% compounded monthly. You can make additional monthly payments to savings, also earning 6%. How much must you save each month to get the appropriate down payment?

This time we have three different cash flows. 1) The money from the initial $20,000, 2) the monthly savings, and 3) the mortgage payments.
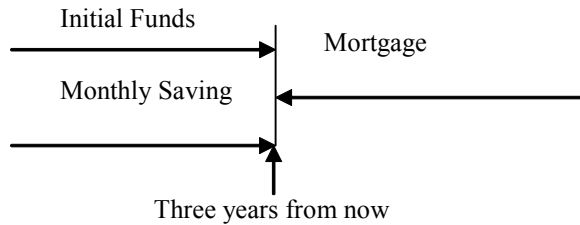
The money from the initial $20,000 is just a future value of a single amount; we can solve for this value now.

The monthly saving is a future value of an annuity. We can't solve for the amount of the payments until we know how much the future value must be.

The mortgage payment is a little trickier. The mortgage payments start three years from now and go for the next 30 years. Even though this is in the future, this is still a *present value problem*. We are converting multiple payments to a single value at the start, which is a 'present value of an annuity' problem.

Since we can figure out the mortgage payments from the restriction on income, we have enough information to solve for the amount of the loan.

Our strategy will be to determine how much you can borrow. From the price of the house we can then determine the amount of the down payment you need. We subtract away the initial $20,000 from the down payment and then we can determine how much you need to save.

Determining the size of the home loan. The mortgage is paid monthly, and so we convert the variables to a monthly basis and solve for the present value. We input:

        N = 360
        I/Y = 0.75
        PV = ?
        PMT=1,250
        FV = 0

The PV = -155,352, which is the amount you can borrow. This leaves you with a down payment of $64,648.

Determining the future value of the initial savings. We convert the variables to a monthly basis and solve for the future value. We input:

    N = 36
    I/Y = 0.5
    PV = -20,000
    PMT=0
    FV = ?

The FV = 23,934. Subtracting this from the needed down payment means that the family must raise $40,714 from monthly saving.

Determine the amount to save each month. This is a future value problem where we are determining the payment. We convert the variables to a monthly basis and input:

    N = 36
    I/Y = 0.5
    PV = 0
    PMT=?
    FV = 40,714

The answer is PMT = -1,035 which means that you would need to save $1,035 per month.