Sec. 7.7: Fundamental Matrices

MATH 351

California State University, Northridge

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Suppose that $x^{(1)}(t), \ldots, x^{(n)}(t)$ form a fundamental set of solutions for the equation

$$x' = P(t)x$$

on some interval $\alpha < t < \beta$. Then the matrix

$$\Psi(t) = (x^{(1)}(t), \ldots, x^{(n)}(t))$$

is said to be a fundamental matrix for the system (1).

**Note:** a fundamental matrix is nonsingular since its columns are linearly independent vectors.
Example 1

Find a fundamental matrix for the system

\[ x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x. \]  

(3)

Solution:
The solution of an IVP in terms of a fundamental matrix

The general solution of (1) is

\[ x = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t) = \Psi(t)c, \]  

(4)

where \( c = (c_1, \cdots, c_n)^T \).

For an IVP of Eq. (1) and the initial condition

\[ x(t_0) = x^0, \]  

(5)

where

- \( t_0 \) is a given point in \( \alpha < t < \beta \).
- \( x_0 \) is a given initial vector.

We need to determine the vector \( c \) using (5) as follows

\[ \Psi(t_0)c = x^0 \implies c = \Psi^{-1}(t_0)x^0. \]  

(6)

Then

\[ x = \Psi(t)\Psi^{-1}(t_0)x^0 \]  

(7)

is the solution of the IVP (1) and (5).

**Note:** \( \Psi' = P(t)\Psi(t) \).
The fundamental matrix $\Phi(t)$

Recall that

**Theorem 7.4.4**

Let

$$e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

further, let $x^{(1)}, \ldots, x^{(n)}$ be the solutions of the system (1) that satisfy the initial conditions

$$x^{(1)}(t_0) = e^{(1)}, \ldots, x^{(n)}(t_0) = e^{(n)},$$

respectively, where $t_0$ is any point in $\alpha < t < \beta$. Then $x^{(1)}, \ldots, x^{(n)}$ form a fundamental set of solutions of the system (1).

The fundamental matrix $\Phi(t)$: $\Phi(t) = (x^{(1)}(t), \ldots, x^{(n)}(t))$.

Note: we use $\Psi$ to denote an arbitrary fundamental matrix.
The fundamental matrix $\Phi(t)$ cont’d

Note:

$$\Phi(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I. \quad (9)$$

Remark: The solution of the IVP (1) and (5) is

$$x = \Phi(t)\Phi^{-1}(t_0)x^0$$

$$= \Phi(t)x^0. \quad (10)$$

Note: If the fundamental matrix $\Phi(t)$ has been determined, then the solution for each set of initial conditions can be found simply by matrix multiplication, as indicated by Eq. (10).

- The fundamental matrix $\Phi(t)$ represents a transformation of the initial condition $x^0$ into the solution $x(t)$ at an arbitrary time $t$.

- $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$. 

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Example 2

For the system (3)

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x$$

in Example 1, find the fundamental matrix $\Phi(t)$ such that $\Phi(0) = I$.

Solution:
The Matrix $\exp(At)$

Recall that the solution of the scalar IVP

$$x' = ax, \quad x(0) = x_0$$  \hspace{1cm} (12)

Where $a$ is a constant, is _______________________

Now consider the corresponding IVP for an $n \times n$ system,

$$x' = Ax, \quad x(0) = x^0,$$ \hspace{1cm} (13)

where $A$ is a constant matrix. Its solution is

_____________________

The comparison of these two solutions suggests that the matrix $\Phi(t)$ might have an exponential character!
The Matrix \( \exp(At) \) cont’d

The scalar exponential function \( \exp(\alpha t) \) can be represented by the power series

\[
\exp(\alpha t) = 1 + \sum_{n=1}^{\infty} \frac{\alpha^n t^n}{n!},
\]

(14)

which converges for all \( t \).

Now, replace the scalar \( \alpha \) by the \( n \times n \) constant matrix \( A \) and consider the corresponding series

\[
I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^n t^n}{n!} + \cdots.
\]

(15)

Thus, the series (15) defines as its sum a new matrix (denoted by \( \exp(At) \)); that is

\[
\exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}
\]

(16)

\[
\frac{d}{dt}[\exp(At)] = \exp(At)
\]

\[
\exp(At)|_{t=0} = I
\]
\( \Phi(t) = \exp(At) \)

Recall that the fundamental matrix \( \Phi(t) \) satisfies the same IVP as \( \exp(At) \), namely

\[
\Phi' = A\Phi, \quad \Phi(0) = I.
\]  

(17)

Then, by the uniqueness part of Theorem 7.2.1 (extended to matrix differential equations), we conclude that

\[
\Phi(t) = \exp(At).
\]

Therefore, the solution of the IVP (13) is in the form of

\[
x = \exp(At)x^0.
\]  

(18)

**Note:** The matrix function \( \exp(At) \) does have the properties we associate with the exponential function. (See Problem 15.)
**Diagonalizable Matrices**

- **coupled system:** some or all of the equations involve more than one—typically—of the unknown variables. (must be solved *simultaneously*.)

- **uncoupled system:** each equation involves only a single variable. (can be solved *independently* of all the others.)

This suggests that

- transform a system of equations into an equivalent **uncoupled** system in which each equation contains **only one** unknown variable.

- transform the coefficient matrix $A$ into a **diagonal** matrix.
Suppose that the $n \times n$ matrix $A$ has a full set of $n$ linearly independent eigenvectors. (This can be the case if the eigenvalues of $A$ are all different, or if $A$ is Hermitian.)

$\xi^{(1)}, \ldots, \xi^{(n)}$ : eigenvectors

$\lambda_1, \ldots, \lambda_n$ : the corresponding eigenvalues.

Now to form the matrix $T$ whose columns are the eigenvectors,

$$T = \begin{pmatrix}
\xi_1^{(1)} & \cdots & \xi_1^{(n)} \\
\vdots & \ddots & \vdots \\
\xi_n^{(1)} & \cdots & \xi_n^{(n)}
\end{pmatrix}$$  \hspace{1cm} (19)

**Note:** the columns of $T$ are linearly independent vectors, $\det T \neq 0 \implies T^{-1}$ exists.

$AT =$
Diagonalizable Matrices Cont’d

\[ AT = TD \quad (20) \]
\[ \downarrow \]
\[ T^{-1}AT = D. \quad (21) \]

**Similarity Transformation:** the process shown in (21) which can be used to transform \( A \) into a diagonal matrix if the eigenvalues and eigenvectors of \( A \) are known.

**Note:**
- (21) means that \( A \) is similar to the diagonal matrix \( D \). (\( A \) is diagonalizable.)
- The similarity transformation leaves the eigenvalues of \( A \) unchanged and transform its eigenvectors into the coordinate vectors \( e^{(1)}, \ldots, e^{(n)} \).

**Remark:** If \( A \) has fewer than \( n \) linearly independent eigenvectors, then there is no matrix \( T \) such that \( T^{-1}AT = D \). (\( A \) is not similar to a diagonal matrix and is not diagonalizable.)
Consider the matrix

\[
A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix},
\]

find the similarity transformation matrix \( T \) and show that \( A \) can be diagonalized.

**Solution:**
Solving $x' = Ax$ by diagonalizing the coefficient matrix $A$

For the system

$$x' = Ax,$$  \hspace{1cm} (23)

where $A$ is a constant matrix.

- In Sec. 7.5 and 7.6,

- Now, to solve the system based on diagonalizing the coefficient matrix $A$.
  
  If $A$ has a full set of $n$ linearly independent eigenvectors,
  - $\xi^{(1)}, \cdots, \xi^{(n)}$ eigenvectors of $A$
  - $r_1, \cdots, r_n$ the corresponding eigenvalues
  - $T = (\xi^{(1)}, \cdots, \xi^{(n)})$ the transformation matrix
  
  Define a new dependent variable $y$ by

$$x = Ty \implies$$ \hspace{1cm} (24)
A fundamental matrix for the system $y' = Dy$ is the diagonal matrix

$$Q(t) = \exp(Dt) =$$ (25)

A fundamental matrix for the system $x' = Ax$ is then found from $Q$ by the transformation $x = Ty$

$$\Psi = TQ =$$ (26)
Example 4

Solve the system of DE (in Example 1)

\[ x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x \]  \hspace{1cm} (27)

by diagonalizing the coefficient matrix.

Solution: