The Wave Equation

Goal: Linear and rotational physics allow us to incorporate photorealism into the motion of rigid bodies, simulating more complex physical phenomena (*i.e*, fluid motion, the simulation of fire and smoke, or cloth motion) involve the solution of PDEs. In this lecture we use Newton's second law to derive *the wave equation*, a simple PDE that governs a wide range of physical phenomena and will lead us into a number of computational methods valuable for creating photorealistic animations.

I. Vibrating String

In order to derive the wave equation, we consider a vibrating *flexible* string:

- L - length (ends fix at $x = 0$ and $x = L$)
- σ – *constant* linear density (mass per unit length)
- τ – tension stretching the string
- $f(x, t)$ – load on the string (positive in downward direction)
- we consider motion on the vertical xy-plane (*i.e.*, the string is fix at the ends and moves only up and down)

We want to determine the displacement $y(x, t)$ under the assumptions:

1. the slope is small, $|\partial y/\partial x| \ll 1$, (*i.e.*, the string is tight)
2. only force acting on *cross sections* of string is τ which is tangential to the curve y

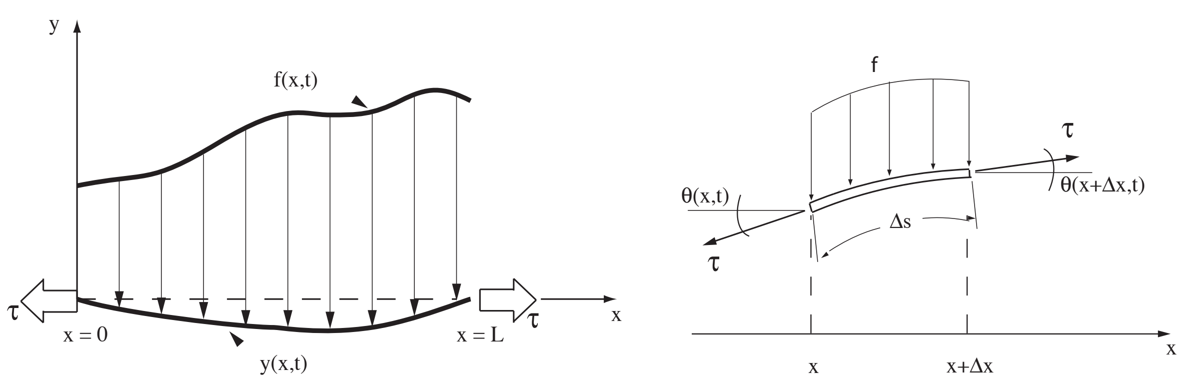

Figure 1: Left: loaded vibrating string, right: string element.

we now consider a piece of the string extending from x to $x + \Delta x$, and apply Newton's second law to it,

$$\tau \sin\theta(x + \Delta x, t) - \tau \sin\theta(x, t) - f(x + \alpha\Delta x, t)\Delta x = \sigma \Delta s \frac{\partial^2 y}{\partial t^2}(x + \beta\Delta x, t), \tag{1}$$

where:

- $\Delta s = \Delta x / \cos\theta$ – arclength $\quad \Rightarrow \quad$ $\sigma\Delta s$ – mass of the string element
- $0 \leq \alpha \leq 1$ is s.t. $f(x + \alpha\Delta x, t)$ is the average value of $f(x, t)$ over the interval $[x, x + \Delta x]$
  $$\Rightarrow \quad f(x + \alpha\Delta x, t)\Delta x \text{ – total load on string element}$$
- $x + \beta\Delta x$ – location of the mass center

Observation: for $\theta \ll 1$ (a reasonable assumption for a tight string), we have

$$\begin{aligned}
\sin\theta &= \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots \approx \theta, \\
\cos\theta &= 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \cdots \approx 1, \\
\tan\theta &= \theta + \frac{1}{3!}\theta^3 + \frac{2}{15}\theta^5 + \cdots \approx \theta,
\end{aligned}$$

so, we can approximate:

$$\frac{\partial y}{\partial x} = \tan\theta \approx \sin\theta \qquad \text{and} \qquad \Delta s = \frac{\Delta x}{\cos\theta} \approx \Delta x,$$

and write (1) as

$$\tau \frac{\frac{\partial y}{\partial x}(x + \Delta x, t) - \frac{\partial y}{\partial x}(x, t)}{\Delta x} - f(x + \alpha\Delta x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x + \beta\Delta x, t), \tag{2}$$

and letting $\Delta x \to 0$, we arrive at

$$\tau \frac{\partial^2 y}{\partial x^2}(x, t) - f(x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x, t). \tag{3}$$

If the load on the string is due to gravity, then $f(x, t) = \sigma g = constant$, and we can write

$$\tau \frac{\partial^2 y}{\partial x^2}(x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x, t) + \sigma g, \tag{4}$$

and if the effect of g is negligible (**Q:** is it? – HW), letting $c = \sqrt{\frac{\tau}{\sigma}}$, we arrive at the wave equation

$$y_{tt} = c^2 y_{xx}. \tag{5}$$

II. D'Alambert Solution. We now seek a solution of the wave equation by introducing the change of variables

$$\xi = x - ct \qquad \text{and} \qquad \eta = x + ct, \tag{6}$$

and expressing the partial derivatives with respect to x and t respectively as

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},\\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta},\end{aligned}$$

the wave equation becomes

$$\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right) y = c^2 \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) y,$$

which reduces to

$$y_{\xi\eta} = 0. \tag{7}$$

Question: How? **Answer:** next HW

This equation can be integrated to obtain, first

$$y_\xi = \int 0\, d\eta = 0 + A(\xi) \quad \Rightarrow \quad y = \int A(\xi)\, d\xi = F(\xi) + G(\eta),$$

and undoing the change of variables, we get a *general solution* for the wave equation.

$$y(x,t) = F(x - ct) + G(x + ct) \tag{8}$$

Remark: notice that nothing has been assumed about F and G, which means that any arbitrary choice will do... Try it (HW).

Example: consider the initial value problem for an infinite string

$$\begin{gathered} y_{tt} = c^2 y_{xx}, \quad -\infty < x < \infty, \qquad 0 < t < \infty \\ y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad -\infty < x < \infty. \end{gathered}$$

Using D'Alambert's solution, we write

$$\begin{aligned} y(x,0) &= f(x) &&= F(x) + G(x),\\ y_t(x,0) &= g(x) &&= -c\, F'(x) + c\, G(x), \end{aligned}$$

integrating the second of these equations, we obtain

$$\int_0^x g(\xi)\, d\xi = -c\, F(x) + c\, F(0) + c\, G(x) - c\, G(0),$$

and combining this with the first of the above, we can solve for $F(x)$ and $G(x)$

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\xi)\, d\xi + \frac{F(0) - G(0)}{2},$$
$$G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\xi)\, d\xi - \frac{F(0) - G(0)}{2}.$$

So replacing x with $x - ct$ in the first of these and with $x + ct$ in the second, we can write

$$\begin{aligned} y(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{f(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\, d\xi + \frac{F(0)-G(0)}{2} \\ &\quad + \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(\xi)\, d\xi - \frac{F(0)-G(0)}{2}, \end{aligned}$$

or

$$y(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi. \tag{9}$$

III. An Application: Water Waves

Consider plane water waves in water of depth $h(x)$. If the wavelength is much greater than h (true for ocean waves and certain shallow water waves), the governing equations are

$$\begin{aligned} u_t + u u_x &= -g\eta_x, \\ [u(\eta + h)]_x &= -\eta_t, \end{aligned}$$

where

- $u(x,t)$ – velocity of the *column* of water
- $\eta(x,t)$ – free-surface elevation relative to undisturbed water level
- g – acceleration of gravity

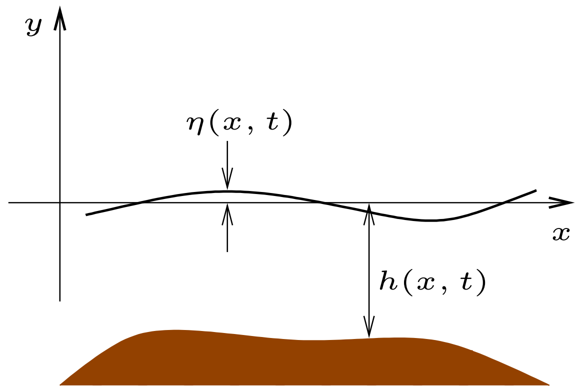

Figure 2: water wave

For small amplitude waves, $uu_x \ll u_t, g\eta_x$, and $\eta \ll h$. Then, one can show (HW) that η satisfies,

$$g(h\eta_x)_x = \eta_{tt}$$

and if $h(x)$ is constant (flat ocean floor),

$$c^2\eta_{xx} = \eta_{tt}$$

Question: what is c in this case?