The Wave Equation

Goal: Linear and rotational physics allow us to incorporate photorealism into the motion of rigid bodies, simulating more complex physical phenomena (i.e., fluid motion, the simulation of fire and smoke, or cloth motion) involve the solution of PDEs. In this lecture we use Newton’s second law to derive the wave equation, a simple PDE that governs a wide range of physical phenomena and will lead us into a number of computational methods valuable for creating photorealistic animations.

I. Vibrating String

In order to derive the wave equation, we consider a vibrating flexible string:

- L - length (ends fix at $x = 0$ and $x = L$)
- $\sigma$ – constant linear density (mass per unit length)
- $\tau$ – tension stretching the string
- $f(x, t)$ – load on the string (positive in downward direction)
- we consider motion on the vertical $xy$-plane (i.e., the string is fix at the ends and moves only up and down)

We want to determine the displacement $y(x, t)$ under the assumptions:

1. the slope is small, $|\partial y/\partial x| \ll 1$, (i.e., the string is tight)
2. only force acting on cross sections of string is $\tau$ which is tangential to the curve $y$

![Figure 1: Left: loaded vibrating string, right: string element.](image-url)
we now consider a piece of the string extending from \( x \) to \( x + \Delta x \), and apply Newton’s second law to it,

\[
\tau \sin \theta(x + \Delta x, t) - \tau \sin \theta(x, t) - f(x + \alpha \Delta x, t) \Delta x = \sigma \Delta s \frac{\partial^2 y}{\partial t^2}(x + \beta \Delta x, t),
\]

(1)

where:

- \( \Delta s = \Delta x / \cos \theta \) – arclength \( \Rightarrow \) \( \sigma \Delta s \) – mass of the string element
- \( 0 \leq \alpha \leq 1 \) is s.t. \( f(x + \alpha \Delta x, t) \) is the average value of \( f(x, t) \) over the interval \([x, x + \Delta x] \) \( \Rightarrow f(x + \alpha \Delta x, t) \Delta x \) – total load on string element
- \( x + \beta \Delta x \) – location of the mass center

**Observation:** for \( \theta \ll 1 \) (a reasonable assumption for a tight string), we have

\[
\sin \theta = \theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \cdots \approx \theta,
\]

\[
\cos \theta = 1 - \frac{1}{2} \theta^2 + \frac{1}{4!} \theta^4 + \cdots \approx 1,
\]

\[
\tan \theta = \theta + \frac{1}{3!} \theta^3 + \frac{2}{15} \theta^5 + \cdots \approx \theta,
\]

so, we can approximate:

\[
\frac{\partial y}{\partial x} = \tan \theta \approx \sin \theta \quad \text{and} \quad \Delta s = \frac{\Delta x}{\cos \theta} \approx \Delta x,
\]

and write (1) as

\[
\tau \frac{\partial y}{\partial x}(x + \Delta x, t) - \frac{\partial y}{\partial x}(x, t) - f(x + \alpha \Delta x, t) \Delta x = \sigma \frac{\partial^2 y}{\partial t^2}(x + \beta \Delta x, t),
\]

(2)

and letting \( \Delta x \to 0 \), we arrive at

\[
\tau \frac{\partial^2 y}{\partial x^2}(x, t) - f(x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x, t).
\]

(3)

If the load on the string is due to gravity, then \( f(x, t) = \sigma g = \text{constant} \), and we can write

\[
\tau \frac{\partial^2 y}{\partial x^2}(x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x, t) + \sigma g,
\]

(4)

and if the effect of \( g \) is negligible (Q: is it? – HW), letting \( c = \sqrt{\frac{\tau}{\sigma}} \), we arrive at the wave equation

\[
y_{tt} = c^2 y_{xx}.
\]

(5)
II. D’Alambert Solution. We now seek a solution of the wave equation by introducing the change of variables

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct,$$

and expressing the partial derivatives with respect to $x$ and $t$ respectively as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t},$$

the wave equation becomes

$$\left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) \left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta}\right) y = c^2 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) y,$$

which reduces to

$$y_{\xi\eta} = 0.$$  \hspace{1cm} (7)

**Question:** How? **Answer:** next HW

This equation can be integrated to obtain, first

$$y_\xi = \int 0 \, d\eta = 0 + A(\xi) \implies y = \int A(\xi) \, d\xi = F(\xi) + G(\eta),$$

and undoing the change of variables, we get a general solution for the wave equation.

$$y(x, t) = F(x - ct) + G(x + ct)$$  \hspace{1cm} (8)

**Remark:** notice that nothing has been assumed about $F$ and $G$, which means that any arbitrary choice will do... Try it (HW).

**Example:** consider the initial value problem for an infinite string

$$y_{tt} = c^2 y_{xx}, \quad -\infty < x < \infty, \quad 0 < t < \infty$$

$$y(x, 0) = f(x), \quad y_t(x, 0) = g(x), \quad -\infty < x < \infty.$$

Using D’Alambert’s solution, we write

$$y(x, 0) = f(x) = F(x) + G(x),$$

$$y_t(x, 0) = g(x) = -c F'(x) + c G(x),$$
integrating the second of these equations, we obtain

\[ \int_0^x g(\xi) \, d\xi = -c F(x) + c F(0) + c G(x) - c G(0), \]

and combining this with the first of the above, we can solve for \( F(x) \) and \( G(x) \)

\[ F(x) = \frac{f(x)}{2} - \frac{1}{2c} \int_0^x g(\xi) \, d\xi + \frac{F(0) - G(0)}{2}, \]
\[ G(x) = \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(\xi) \, d\xi - \frac{F(0) - G(0)}{2}. \]

So replacing \( x \) with \( x - ct \) in the first of these and with \( x + ct \) in the second, we can write

\[ y(x, t) = F(x - ct) + G(x + ct) \]
\[ = \frac{f(x - ct)}{2} - \frac{1}{2c} \int_0^{x-ct} g(\xi) \, d\xi + \frac{F(0) - G(0)}{2} \]
\[ + \frac{f(x + ct)}{2} + \frac{1}{2c} \int_0^{x+ct} g(\xi) \, d\xi - \frac{F(0) - G(0)}{2}, \]

or

\[ y(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) \, d\xi. \tag{9} \]

III. An Application: Water Waves

Consider plane water waves in water of depth \( h(x) \). If the wavelength is much greater than \( h \) (true for ocean waves and certain shallow water waves), the governing equations are

\[ u_t + uu_x = -g \eta_x, \]
\[ [u(\eta + h)]_x = -\eta_t, \]

where

- \( u(x, t) \) – velocity of the column of water
- \( \eta(x, t) \) – free-surface elevation relative to undisturbed water level
- \( g \) – acceleration of gravity
For small amplitude waves, \( uu_x \ll u_t, g\eta_x \) and \( \eta \ll h \). Then, one can show (HW) that \( \eta \) satisfies,

\[
g(h\eta_x)_x = \eta_{tt}
\]

and if \( h(x) \) is constant (flat ocean floor),

\[
c^2 \eta_{xx} = \eta_{tt}
\]

**Question:** what is \( c \) in this case?