

Homework 2 – The wave equation

Due: Thurs., April 30, 2009

Directions: solve the following problems. You can work with others and discuss the problems, but each student must write his/her own, independent solution. If you are unsure about what i mean by this, please ask!

What to turn in? Sort the solutions of the assigned problems –by chapter and problem number, staple the pages together, and write your name, student ID, MATH496 and HW # 1 in the front page. For problem # 3, turn in a printout of your code and a plot of the initial conditions and final state in the same graph.

Problems:

Problem 1. If spherical symmetry is present so that u depends only on ρ and t (for spherical coordinates ρ , θ , and ϕ), then the wave equation, $c^2\Delta u = u_{tt}$ becomes

$$\frac{c^2}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) = \frac{\partial^2 u}{\partial t^2}.$$

Show that this equation can be expressed as

$$c^2 \frac{\partial^2}{\partial \rho^2} (\rho u) = \frac{\partial^2}{\partial t^2} (\rho u),$$

and derive the general solution

$$u(\rho, t) = \frac{1}{\rho} [F(\rho - ct) + G(\rho + ct)],$$

where F and G are arbitrary twice differentiable functions.

Problem 2. Show that, like the wave equation, the given PDEs are hyperbolic and find its general solution by introducing the suggested change of variables.

(a) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$; $\xi = x - y$, $\eta = 3x - y$.

(b) $u_{xx} - 4u_{xy} - 5u_{yy} = 0$; $\xi = x - y$, $\eta = 5x + y$.

(c) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$; $\xi = ax + by$, $\eta = cx + dy$, you'll need to determine a, b, c , and d in this one.

Problem 3. Use D'Alambert's formula to find the solution of the wave equation, $y_{tt} = c^2 y_{xx}$, with initial conditions

$$u(x, 0) = f(x) = 0, \quad u_t(x, 0) = g(x) = \begin{cases} 1 & \text{if } |x| < h \\ 0 & \text{if } |x| > h. \end{cases}$$

Problem 4. Write a matlab script `wave.m` and a function file `F.m` that calculates and plots the solution of the **Problem 3** above over the interval $t \in [0, 10]$, use $c = h = 1$. **Extra credit:** create an animation of your solution.

Problem 5. Refer to the for [lecture notes](#) for the wave equation (part III). Consider the governing equations for shallow water waves

$$\begin{aligned}u_t + uu_x &= -g\eta_x, \\ [u(\eta + h)]_x &= -\eta_t,\end{aligned}$$

show that if the term uu_x is negligible with respect to the terms u_t and $g\eta_x$ and η can be neglected with respect to h , then η satisfies

$$g(h\eta_x)_x = \eta_{tt}$$

or, if $h(x)$ is constant (flat bottom),

$$c^2\eta_{xx} = \eta_{tt}.$$

What is c in this case?