Homework # 2

Probability and Information Theory

Due: Fri. Oct. 12, 2018

Goal: Reinforce common principles of probability theory and information theory that are applicable to machine learning

Directions:

Solve the problems listed below.

Submit a file named `math382_hw2_yx.pdf` (where `y` and `x` stand for your initials in reverse order) with your solutions here .

You can work with others and discuss the problems, but each student must write his/her own, independent solution. If you are unsure what i mean by this, please ask!

Problem 1. A box contains three marbles, one red, one blue, and one green. Describe the sample space of each of the following experiments:

(a) First draw a marble from the box, record its color, replace the marble in the box, and draw a marble again.

(b) Draw two marbles without replacing the first one.

Problem 2. 60 % of students at a certain school wear neither a ring nor a bracelet, 20 % wear a ring, and 30 % wear a bracelet. If one student is chosen randomly, what is the probability that the student is wearing:

(a) a ring or a bracelet?

(b) a ring and a bracelet?

Problem 3. A pair of fair dice are rolled one at a time. What is the probability that the second die lands on a higher value that the first?

Problem 4. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Problem 5. A total of 45 % of voters in a certain city classify themselves as Independents, 30 % as Liberals, and 25 % as Conservatives. In a recent election, 37 % of the Independents, 60 % of the Liberals, and and 55 % of the Conservatives voted. A voter is chosen at random. Given that this person voted in the local election, what is the probability that s/he is

(a) an Independent?

(b) a Liberal?

(c) a Conservative?

(d) What fraction of voters participated in the election?

Problem 6. Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(a) the maximum value to appear in the two rolls?

(b) the minimum value to appear in the two rolls?

(c) the sum of the two rolls

(d) the value of the first roll, minus the value of the second roll?

Problem 7. Suppose that the distribution function of X is given by:

$$F(a) = \begin{cases} 0 & \text{if } a < 0 \\ \frac{a}{4} & \text{if } 0 \le a < 1 \\ \frac{1}{2} + \frac{a-1}{4} & \text{if } 1 \le a < 2 \\ \frac{11}{12} & \text{if } 2 \le a < 3 \\ 1 & \text{if } a \ge 3 \end{cases}$$

(a) Find $P\{X = i\}$, $i = 1, 2, 3$.

(b) Find $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\}$.

Problem 8. The lifetime in hours of an electronic part is a random variable having probability density function $f(x) = xe^{-x}$, $x \ge 0$. Find the expected lifetime of the part.

Problem 9. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find a and b if $\mathbb{E}[X] = \frac{3}{5}$.

Problem 10. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, find:

(a) $P\{X > 5\}$;

(b) $P\{4 < X < 16\}$;

(c) $P\{X < 8\}$;

(d) $P\{X < 20\}$;

(e) $P\{X > 16\}$.

Problem 11. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \quad 0 < x < 1, \;\; 0 < y < 2.$$

(a) Verify $f(x,y)$ is a joint probability density function.

(b) Compute the probability density function of X.

(c) Find $P\{X > Y\}$.

Problem 12. Let X and Y be independent random variables, both being equally likely to take in any of the values $1, 2, \ldots, m$. Show that

$$\mathbb{E}[|X - Y|] = \frac{(m-1)(m+1)}{3m}$$

Problem 13. A pair of fair dice is rolled. Let

$$X = \begin{cases} 1 & \text{if the sum of the dice is 6} \\ 0 & \text{otherwise} \end{cases}$$

and let Y equal the value fo the first die. Compute

(a) $H(Y)$;

(b) $H_Y(X)$;

(c) $H(X,Y)$.

Problem 14. For any random variable X and function f, show that $H(f(X)) \leq H(X)$.