Children, Toys, Joggers, and Dogs

Goal: In this lecture we extend our discussion of the Tractrix to find the trajectories of other bodies moving subject to other constraints.

Problem I. The Child and the Toy: A child moves along a curve \( r_c = (X(t), Y(t)) \) while dragging a toy attached to a straight rigid bar. What is the trajectory, \( r_t = (x(t), y(t)) \), of the toy?

Solution:

\( a \) – length of the bar

\( r_c = (X(t), Y(t)) \) – child’s position vector (known)

\( r_t = (x(t), y(t)) \) – Toy’s position vector (unknown)

\( v_c = (\dot{X}(t), \dot{Y}(t)) \) – child’s velocity

\( v_t = (\dot{x}(t), \dot{y}(t)) \) – toy’s velocity

\( \alpha \) – \( v_c \angle v_t \)

Observations:

1. The length of the bar (distance between the child and the toy) is constant

\[ (X - x)^2 + (Y - y)^2 = a^2 \]  \( \text{(1)} \)

2. The toy’s motion is in the direction of the bar,

\[ \begin{pmatrix} X - x \\ Y - y \end{pmatrix} = \lambda \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \quad \lambda > 0 \]  \( \text{(2)} \)

3. The speed of the toy depends on the direction of the velocity of the child, e.g., if the child moves in a circle of radius \( a \), the toy won’t move. The magnitude of \( v_t \) is the projection of \( v_c \) onto the bar
We now insert equation (2) into (1) and obtain

\[ a^2 = \lambda^2 (\dot{x}^2 + \dot{y}^2) \quad \Rightarrow \quad \lambda = \frac{a}{\sqrt{\dot{x}^2 + \dot{y}^2}} \]  

(3)

so we arrive at the system of (coupled) ODEs

\[ \frac{a}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} X - x \\ Y - y \end{pmatrix} \]  

(4)

We’ll use MATLAB ODE45 to solve this system. Here is a brief explanation of what ODE45 does:

**ODE45** Solve non-stiff differential equations, medium order method.

\[ [\text{TOUT}, \text{YOUT}] = \text{ODE45}(\text{ODEFUN}, \text{TSPAN}, \text{Y0}) \]  

with TSPAN = [T0 TFINAL] integrates the system of differential equations \( y' = f(t,y) \) from time T0 to TFINAL with initial conditions Y0. ODEFUN is a function handle. For a scalar \( T \) and a vector \( Y \), ODEFUN(T,Y) must return a column vector corresponding to \( f(t,y) \). Each row in the solution array YOUT corresponds to a time returned in the column vector TOUT. To obtain solutions at specific times T0,T1,...,TFINAL (all increasing or all decreasing), use TSPAN = [T0 T1 ... TFINAL].

**Recall:** Our goal is to solve for \( r_t = (x(t), y(t)) \), and so far we have

\[ \frac{a}{\|r_t\|} \dot{r}_t = r_c - r_t \]  

(5)

not in the form \( y' = f(t, y) \) as we need! Something needs to be done . . .

- Our system indicates that the velocity of the toy points in the direction of \( r_c - r_t = (X - x, Y - y) \), so we’ll seek a unit vector in that direction

\[ w = \frac{r_c - r_t}{\|r_c - r_t\|} \]  

(6)

- We also notice that \( \|v_t\| = \|v_c\| \cos \alpha \), and that \( v_c \cdot w = \|v_c\| \|w_c\| \cos \alpha \), or

\[ \|v_t\| = v_c \cdot w \]  

(7)

We now have the direction and magnitude of \( v = \dot{r} \), so we can write

\[ \dot{r} = (v_c \cdot w)w \]  

(8)

where \( w \) is a function of \( t \) and \( y \). So now we have a system of the form \( y' = f(t, y) \) and can use ODE45. We’ll need:
1. a function $f(t, y)$ (we’ll actually use $f(t, z)$ to define it so as to avoid confusion),

```matlab
function zs = f(t, z)
%
[X Xs Y Ys] = child(t);
v = [Xs; Ys];
w = [X-z(1); Y-z(2)];
w = w/norm(w);
zs = (v'*w)*w;
```

2. another function to find the position and velocity of the child, let’s take a circle

```matlab
function [X, Xs, Y, Ys] = child(t);
%
X = 5*cos(t); Y = 5*sin(t);
Xs = -5*sin(t); Ys = 5*cos(t);
```

3. and a MATLAB routine that sets the initial conditions, calls ODE45, plot, etc...

```matlab
% main1.m
y0 = [10 0]’;
[t y] = ode45(’f’, [0 100], y0);
clf; hold on;
axis([-6 10 -6 10]);
axis(’square’);
plot(y(:,1),y(:,2));

clf; hold on;
axis([-6 10 -6 10]);
axis(’square’);
plot(y(:,1),y(:,2));

% main1.m
y0 = [10 0]’;
[t y] = ode45(’f’, [0 100], y0);
clf; hold on;
axis([-6 10 -6 10]);
axis(’square’);
plot(y(:,1),y(:,2));

initial conditions are $r_c = (5, 0)$ and $r_t = (10, 0)$, and we solve from $t = 0$ to $t = 100$. 
Problem II. The Jogger and the Dog: a jogger runs along a trail on a plane, \( r_j = (X(t), Y(t)) \), a dog sees him and starts running towards him at constant speed \( w \). Find the orbit of the dog, \( r_d = (x(t), y(t)) \).

Observations:

1. The dog runs with constant speed: \( \dot{x}^2 + \dot{y}^2 = w^2 \)

2. The velocity vector of the dog is parallel to the difference vector between the position of the jogger and the dog

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \lambda \begin{pmatrix} X-x \\ Y-y \end{pmatrix}, \quad \lambda > 0
\]  \( \tag{9} \)

which leads to

\[
\dot{x}^2 + \dot{y}^2 = w^2 = \lambda^2 \left\| \begin{pmatrix} X-x \\ Y-y \end{pmatrix} \right\|^2 \Rightarrow \lambda = \frac{w}{\left\| \begin{pmatrix} X-x \\ Y-y \end{pmatrix} \right\|} \quad \tag{10} \]

and the system of ODEs

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{w}{\left\| \begin{pmatrix} X-x \\ Y-y \end{pmatrix} \right\|} \begin{pmatrix} X-x \\ Y-y \end{pmatrix} \quad \tag{11} \]

To solve for this system, we’ll use the M-files: \texttt{jogger.m}, \texttt{main2.m}, and \texttt{dog.m}

```matlab
function [zs,isterminal,direction] = dog(t,z,flag);
% global w % w = speed of the dog
X= jogger(t);
h= X-z;
nh= norm(h);
if nargin < 3 | isempty(flag) % normal output
    zs= (w/nh)*h;
else
    switch(flag)
    case 'events' % at norm(h)=0 there is a singularity
        zs= nh-1e-3; % zero crossing at pos_dog=pos_jogger
        isterminal= 1; % this is a stopping event
        direction= 0; % don’t care if decrease or increase
    otherwise
```
error(['Unknown flag: ' flag]);
end
end