Math 151A
Homework #1
Solutions to Selected Problems

October 5, 2001

Section 2.2, Problem 11 (b). Note: this problem doesn't have a unique solution, the solution I present here includes remarks that, I hope, will help you understand the problem better.

Consider the fixed point iteration $x_{n+1} = \frac{5}{x_n^2} + 2$.

(1) Find an interval $[a, b]$ such that the fixed point iteration will converge.

Write the iteration as $x_{n+1} = g(x_n)$, with $g(x) = \frac{5}{x^2} + 2$, by theorem 2.3, we need to find an interval $[a, b]$ such that $|g'(x)| \leq k < 1$ for all $x \in [a, b]$.

Since $|g'(x)| = |\frac{-10}{x^3}|$, it is easy to obtain $|g'(x)| < 1$ for $x \in (-\infty, -10^{\frac{1}{3}}) \cup (10^{\frac{1}{3}}, \infty)$. That means, that if we start the iteration in either of those two intervals, it will converge to a fixed point.

I believe that the problem wants an interval $[a, b] \subset (-\infty, -10^{\frac{1}{3}}) \cup (10^{\frac{1}{3}}, \infty)$ that contains the fixed point itself. When I explained the problem in class, I didn't take this into account, I simply chose an interval, $[3, \infty)$, where we could choose an initial value that will guaranty convergence (not necessarily inside the interval chosen). The book (see solutions in the back) chooses $[2.5, 3]$ which includes the fixed point. The book doesn't tell you how they know the fixed point is there, although there exist some algorithms to determine that, the easiest way would be to plot in the same graph x and $g(x)$, and see (guess) where they cross each other.

(2) Estimate the number of iterations needed to obtain approximations accurate to within 10^{-5}, and perform the calculations.

Using the interval $[2.5, 3]$, we can chose $x_0 = 2.5$. Observe that for this interval, $|g'(x)|$ is maximum at $x = 2.5$, this means that $|g'(x)| \leq k = 0.64 < 1$ for all $x \in [2.5, 3]$. Also, observe that $x_1 = 2.8$

Using corollary 2.4, we need to find n such that:

$e_n = |p - x_n| \leq \frac{k^n}{1-k}\ |x_1 - x_0| < 10^{-5}$

that is:

$\frac{0.64^n}{1-0.64}\ |2.8 - 2.5| < 10^{-5} \Rightarrow n > ln(\frac{10^{-5}(0.36)}{0.3})\frac{1}{ln(0.64)} \approx 25.4.$

Therefore $n = 26$, will guaranty the disired accuracy. This does not mean that such accuracy cannot be attained with less iterations, $n = 26$ is only an estimate that guaranties it. You don't really need to perform the calculations (the 26 iterations), since you don't really know the exact value of p.