Physics 100b
Lecture Notes Spring 2007

Henk Postma
CSU Northridge

(Dated: March 26, 2007)

Lightning over the outskirts of Oradea, Romania, during the August 17 2005 thunderstorm which went on to cause major flash floods over Southern Romania. Author: Mircea Madau `http://www.photo.madau.net`

I. MATHEMATICAL BACKGROUND

We review the most important math and physics material that this class is based on. This chapter is not intended to be a comprehensive text, it is intended to refresh your memory. Please review your material from previous classes if some of the concepts here seem new.

A. Greek letters/symbols

We will use many Greek letters throughout this course, so the Greek alphabet is listed below.

lowercase	uppercase	name	lowercase	uppercase	name
α	A	*alpha*	ν	N	*nu*
β	B	*beta*	ξ	Ξ	*xi*
γ	Γ	*gamma*	o	O	*omicron*
δ	Δ	*delta*	π	Π	*pi*
ϵ	E	*epsilon*	ρ	R	*rho*
ζ	Z	*zeta*	σ	Σ	*sigma*
η	H	*eta*	τ	T	*tau*
θ	Θ	*theta*	υ	Υ	*upsilon*
ι	I	*iota*	ϕ	Φ	*phi*
κ	K	*kappa*	χ	X	*chi*
λ	Λ	*lambda*	ψ	Ψ	*psi*
μ	M	*mu*	ω	Ω	*omega*

B. Algebra

1. Properties of powers

$$\begin{aligned} x^n x^m &= x^{n+m} \\ (x^n)^m &= x^{nm} \\ x^{-n} &= \tfrac{1}{x^n} \\ x^{1/n} &= \sqrt[n]{x} \end{aligned} \tag{1}$$

2. First-order equations and their solutions

Equations where variables are raised to a power not larger than 1

Example:

$$\begin{aligned} &ax + b = c \\ &ax = c - b \\ &x = \tfrac{c-b}{a} \end{aligned} \tag{2}$$

3. Second order equations and their solutions

Equations where the largest power the variables are raised to is 2.

Example:

$$\begin{aligned} &ax^2 + b = c \\ &ax^2 = c - b \\ &x^2 = \tfrac{c-b}{a} \\ &x = \sqrt{\tfrac{c-b}{a}} \quad \vee \quad x = -\sqrt{\tfrac{c-b}{a}} \quad (\text{assuming } \tfrac{c-b}{a} \geq 0) \end{aligned} \tag{3}$$

Example:

$$\begin{aligned} &ax^2 + bx + c = 0 \\ &x = \tfrac{-b+\sqrt{b^2-4ac}}{2a} \quad \vee \quad x = \tfrac{-b-\sqrt{b^2-4ac}}{2a} \quad (\text{assuming } b^2 - 4ac \geq 0) \end{aligned} \tag{4}$$

C. Limits

We use limits to investigate limiting behavior of functions. The symbol ∞ is used for infinity.

Example:

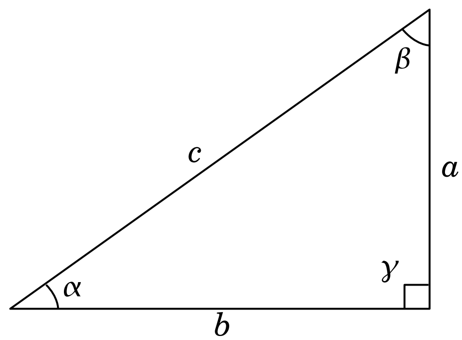
$$\begin{aligned} \lim_{x\to\infty} \tfrac{1}{x} &= 0 \\ \lim_{x\to\infty} e^{-x} &= 0 \\ \lim_{x\to-\infty} e^{x} &= 0 \\ \lim_{x\to\infty} e^{x} &= \infty \end{aligned} \tag{5}$$

D. Trigonometry

Angles can be expressed in degrees or radians. A full circle is 2π radians, or 360 degrees. So converting an angle α from radians to degrees is done by multiplying by 360 and dividing by 2π,

$$\alpha_{\text{degrees}} = \frac{360}{2\pi}\alpha_{\text{radians}}$$

Consider the drawn triangle with sides a, b, c, where the angle between a and b is $\pi/2$. The angles opposite the sides are labeled with their corresponding Greek letters, see section I.A.



Pythagoras' theorem for this triangle states

$$a^2 + b^2 = c^2 \tag{6}$$

The trigonometric functions sine, cosine, and tangent and their inverse are defined as

$$\begin{aligned} \sin\alpha &= a/c & \arcsin a/c &= \alpha \\ \cos\alpha &= b/c & \arccos b/c &= \alpha \\ \tan\alpha &= a/b = \tfrac{\sin\alpha}{\cos\alpha} & \arctan a/b &= \alpha \end{aligned} \tag{7}$$

There are a few special cases, they can be easily confirmed using this triangle

$$\begin{aligned} \sin 0 &= 0 & \cos\pi/2 &= 0 \\ \sin\pi/2 &= 1 & \tan 0 &= 0 \\ \cos 0 &= 1 & \tan\pi/2 &= \infty \end{aligned}$$

The functions sine, cosine and tangent are periodic, which means that there are multiple solutions to trigonometric equations.

> *Example:* We solve $\sin x = 1$, finding $x = \pi/2 \pm n2\pi$, meaning that $\pi/2$ is a solution, but so is $\pi/2 + 2\pi, \pi/2 - 2\pi, \pi/2 + 4\pi, \pi/2 - 4\pi$, etc, i.e. n is a whole number.

E. Vectors

A vector has magnitude *and* direction. They are indicated by an arrow over the symbol, i.e. the vector $\vec{a}$. The magnitude a and direction can be fully specified in several equivalent ways, by components

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

or in row notation $\vec{a} = (a_x, a_y)$.

The length of a vector $\vec{a}$ can be found by the Pythagorean theorem,

$$a = \sqrt{a_x^2 + a_y^2} \tag{8}$$

Example: The length of a vector $(3, 4)$ is $\sqrt{3^2 + 4^2} = 5$.

Vectors can also be defined by their angle and magnitude

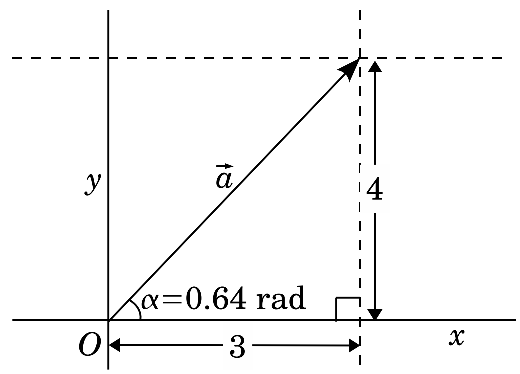
$$\vec{a} = a \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \qquad \alpha = \arctan a_y/a_x$$

Example:
A vector $\vec{a}$ is pointing from the origin to a point on the x, y plane with coordinates $(3, 4)$. The vector can be expressed as

$$\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

or, because $\alpha = \arctan(4/3) = 0.64$ rad, we can also write

$$\vec{a} = 5 \begin{pmatrix} \cos 0.64 \\ \sin 0.64 \end{pmatrix}$$



A vector can have more than 2 components, i.e. vectors in 3D space have 3 components. We then write

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

or in row notation $\vec{a} = (a_x, a_y, a_z)$. We need *two* angles and the length, but we skip that for the time being.

The length of a vector is indictated as $a =\mid \vec{a} \mid$ and is equal to the square root of the sum of the squares of the vector's components, i.e.

$$a =\mid \vec{a} \mid= \sqrt{a_x^2 + a_y^2 + a_z^2}$$

This can be seen as a generalization of the Pythagorean theorem, see what the expression looks like when $a_z = 0$.

The unit vector, indicated with a hat over the symbol, $\hat{a}$, is defined as a vector pointing in the same direction as $\vec{a}$, but with length 1

$$\hat{a} = \frac{\vec{a}}{a}$$

Example: The unit vector corresponding to the vector $\vec{a} = (3,4)$ is

$$\hat{a} = \frac{1}{a}\vec{a} = \frac{1}{\sqrt{3^2+4^2}}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3/5\\4/5\end{pmatrix}$$

In the book, you will find unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. These are vectors of length 1 pointing in the x, y, z direction, respectively. I use $\hat{x}, \hat{y}, \hat{z}$ for these vectors in these notes.

$$\hat{x} = \hat{\imath} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \hat{y} = \hat{\jmath} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \hat{z} = \hat{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Vectors can be expressed in so-called unit vector notation

$$\vec{a} = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = a_x\begin{pmatrix}1\\0\\0\end{pmatrix} + a_y\begin{pmatrix}0\\1\\0\end{pmatrix} + a_z\begin{pmatrix}0\\0\\1\end{pmatrix} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$

1. Vector addition

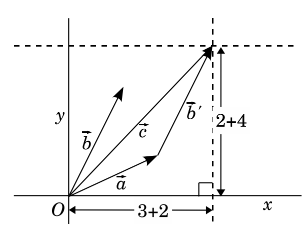
To add two vectors, we add their components

$$\begin{pmatrix}c_x\\c_y\\c_z\end{pmatrix} = \vec{c} = \vec{a} + \vec{b} = \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} + \begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix} = \begin{pmatrix}a_x+b_x\\a_y+b_y\\a_z+b_z\end{pmatrix}$$

Vector addition can be represented graphically by translating the vector $\vec{b}$ over to the end of vector $\vec{a}$, such that the end of $\vec{a}$ coincides with the begin of the translated vector $\vec{b}'$. The sum of the vectors is then a vector from the begin of $\vec{a}$ to the end of $\vec{b}'$.

Example:
We calculate the sum of the vectors $\vec{a}$ and $\vec{b}$. The x components are added to the x components, the y components are added to the y components

$$\vec{a} + \vec{b} = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}c_x\\c_y\end{pmatrix} = \vec{c}$$



2. Vector multiplication: ‘dot’/inner product

The ‘dot’ or inner product between two vectors $\vec{a}$ and $\vec{b}$ is indicated with the central dot

$$\vec{a}\cdot\vec{b} = a_xb_x + a_yb_y + a_zb_z = ab\cos\alpha \tag{9}$$

where α is the angle between the vectors.

Example: The dot product between two vectors $(1, 2, 3)$ and $(4, 5, 6)$ is given by

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 4 + 10 + 18 = 32$$

The length of $\vec{a}$ is $a = \sqrt{14} = 3.7$, while the length of $\vec{b}$ is $b = 8.8$. We can now calculate the angle between the vectors

$$\cos\alpha = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{32}{8.8 \times 3.7} = 0.97 \qquad \rightarrow \qquad \alpha = \arccos 0.97 = 13^\circ$$

3. Special cases of the dot product

The dot product of two *perpendicular* vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\cdot\vec{b} = ab\cos\left(\frac{\pi}{2}\right) = 0$$

The dot product of two *parallel* vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\cdot\vec{b} = ab\cos 0 = ab$$

The dot product of a vector with itself is the square of the length of the vector

$$\vec{a}\cdot\vec{a} = a^2$$

The dot product between unit vectors is

$$\hat{x}\cdot\hat{x} = 1 \qquad , \qquad \hat{x}\cdot\hat{y} = 0$$

II. PHYSICAL BACKGROUND

A. Units

The language of physics is math. However, a major difference between physics and math is that in physics we use units. In math you can always add two numbers together, e.g. to calculate the sum of $12 + 14 = 26$. In physics you can only do these kind of operations when they have the same units. You cannot calculate the sum of 12 N and 14 kg, because they have different units. Similarly, you cannot say which one is bigger.

Sometimes you can convert a quantity from one unit to another unit by using physical law.

Example: Using Newton's 2nd law, we can calculate the gravitational pull on a mass of 14 kg on the surface of the earth, it is 14 N. Now we can compare to, add to, and subtract this number from 12 N.

All quantities in this class are in SI units. It is also known as the MKS system, because distance is in m(eters), mass is in k(ilogram), time in s(econds). We also use some derived units, like N(ewton), T(esla), J(oule) etc. How they relate to the base units of the MKS system can be found by looking at the basic equations that define the quantity.

Example: N(ewton). We use Newton's 2nd law to find how to express N in the MKS system.

$$F = ma \qquad ,$$

the force in N is found by multiplying the mass in kg by the acceleration in m/s^2. The unit N is found by multiplying the corresponding units, so $N = kg\ m/s^2$.

B. Scaling and proportionality

Scaling laws are often important in physics. They allow you to arrive at answers without knowing all details of the equations, using only proportionality, denoted with the symbol $\propto$.

Example: Newton's 2nd law states that the force F required to give a body with mass m an acceleration a

$$F = ma$$

If it takes a force of 1 N to accelarate a body with some mass m, what is the force required to give the same acceleration to a body twice as heavy?
The scaling with m is found by dropping all other multipliers that are not m, in this case a. We find

$$F \propto m$$

F is linearly proportional to m. Increasing m by a factor 2 means F also increases by a factor 2, we find 2 N. We can also find this by dividing both sides of the proportionality by m

$$F/m \propto 1 \quad \text{or} \quad F/m = \text{constant}$$

If m is increased by a factor 2, F must also be increased by a factor 2 for the ratio F/m to remain constant.

Example: Coulomb's law states that the force between two charged particles with charges q_1 and q_2 separated by a distance r is

$$F = \frac{kq_1q_2}{r^2}$$

Knowing that the force is 1 N when the distance is r, what is the force when the distance is doubled?
To find the scaling of F with r, we drop all other multipliers that are not r, in this case k, q_1, q_2. We find the proportionality

$$F \propto \frac{1}{r^2}$$

So if r is doubled, F is reduced by a factor 4. We can also arrive at this answer by noting that because $F \propto 1/r^2$, multiplying both sides by r^2 we see

$$Fr^2 = \text{constant}$$

So if r is increased by a factor 2, F must be decreased by a factor 4 to remain constant, yielding 0.25 N.

C. Forces

Forces are vectors. The unit of force is N(ewton).

1. Net force

When multiple forces are acting on an object, the forces add as vectors, i.e. the x,y, and z components are added together. We distinguish between several different forces, see below. The net force is the resulting force due to all forces acting on a body.

2. Newton's second law

The force $\vec{F}$ required to give a body m an acceleration $\vec{a}$ is equal to

$$\vec{F} = m\vec{a} \tag{10}$$

3. Newton's third law

For every force $\vec{F}_{12}$ that an object 2 exerts on an object 1, there is an opposing force $\vec{F}_{21}$ exactly equal in magnitude that object 1 exerts on object 2,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{11}$$

4. Gravitation

The gravitational attraction $\vec{F}_g$ between two objects is equal to

$$F_g = G\frac{m_1 m_2}{r^2} \tag{12}$$

and is directed towards the other object. We call $\vec{r}_{12}$ the vector pointing from object 2 to object 1, then $\hat{r}_{12}$ is the direction from object 2 to object 1. The gravitational attraction on object 1 is then

$$\vec{F}_{g,12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12} \tag{13}$$

Note that this is the vector notation, and it captures that the force that object 2 exerts on object 1 is pointing in the opposite direction, captured by $-\hat{r}_{12}$.

5. Normal force

The normal force is a force that causes an object not to 'fall' through a surface, and it is always pointing orthogonal to the surface area.

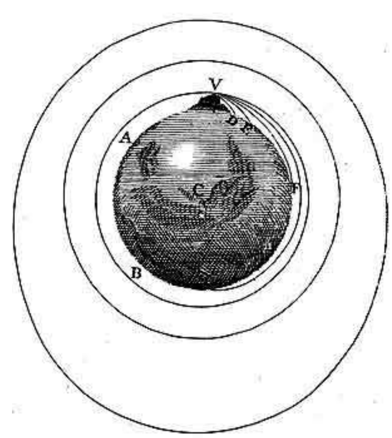
> *Example:* A laptop computer is lying on a table. Gravity is pulling the laptop down. The normal force is exerted on the laptop by the table, exactly equal to gravity, and exactly on the opposite direction. Without the normal force, there would be a net force acting on the laptop, moving it down.

6. Centripetal force

An object with mass m describing a circular orbit with radius r and velocity v requires a force $\vec{F}_c$ acting on it, pointing towards the center of the orbit, to keep it in its orbit.

$$F_c = \frac{mv^2}{r}$$

To illustrate this force, look at Newton's original drawing for this process. If you a rock with a small velocity, standing on top of a mountain, gravity will accelerate the rock towards the surface of the earth. For very small velocities, the trajectory the rock follows is a parabola, and the object hits the earth as indicated. On the scale of the trajectory, the earth can be assumed flat. When the rock is thrown with higher velocity, the curvature of the earth becomes important. The rock will hit the surface of the earth in point D, E, B or A. The trajectory becomes more and more close to a circle. The time spent by the object in orbit becomes longer and longer, because the rock is falling towards the surface, but the surface of the earth is receding from under it. If the rock has just



the right velocity, it will hit you in the back of your head! At that point, the trajectory is a perfect circle, an orbit. In that particular case the force acting on the object is equal to $F_c = mv^2/r$. Note that the centripetal force is the force *required* to keep the object in orbit, and it is pointing towards the center of the earth. Other forces can cause circular orbits as well, electrostatic forces, Lorentz forces etc. We will encounter these in later chapters.

D. Kinematics

The position of an object moving with a constant velocity $\vec{v}$ is given by

$$\vec{r}(t) = \vec{r}_0 + \vec{v}t \qquad \text{or} \qquad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

The position of an object with initial velocity $\vec{v}_0$ that is accelerated with a uniform acceleration $\vec{a}$ is given by

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}t^2\vec{a} \qquad \text{or} \qquad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} + t \begin{pmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{pmatrix} + \frac{1}{2}t^2 \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

while the velocity as a function of time is

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t \qquad \text{or} \qquad \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{pmatrix} + t \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

The kinetic energy of an object with mass m and velocity v is given by

$$E_K = \frac{1}{2}mv^2$$

The units of energy are J(oules).

The momentum of the same object is given by

$$\vec{p} = m\vec{v}$$

The work done by a force $\vec{F}$ along a trajectory $\vec{L}$ is given by

$$W = \vec{F} \cdot \vec{L}$$

Example: You are holding a ball weighing 1 kg 1m above the ground, and drop it. The force is pointing down (in $-\hat{z}$ direction), with magnitude $9.8m$, and the trajectory is also pointing down, with magnitude 1 m, The work done by gravity in moving the ball is equal to

$$W = \vec{F} \cdot \vec{L} = (-F\hat{z}) \cdot (-L\hat{z}) = FL(\hat{z} \cdot \hat{z}) = FL = 9.8 \text{ J}$$

The work done by a force can be converted into kinetic energy. In that case, the work done by the force is equal to the change in kinetic energy, ΔE_K

Example: We calculate the velocity of the ball from the previous example when it hits the ground.

$$\begin{aligned} W &= \Delta E_K \\ 9.8 &= \tfrac{1}{2}mv_{end}^2 - \underbrace{\frac{1}{2}mv_{initial}^2}_{=0} \\ v_{end} &= 4.4 \text{ m/s} \end{aligned} \tag{14}$$

III. ELECTROSTATICS

A. Charge

Charge is either positive or negative, like charges repel, opposite charges attract. Charge is quantised in units of the electron's charge, -1.6×10^{-19} C, and the charge of the electron is negative. The electron is the elementary charged particle. It orbits a positively charged nucleus in an atom, the building block of all materials. See the simplified model of an atom on the right. When we rub objects together, a few electrons may be 'rubbed' off and transferred to the other object, thereby exposing some of the positive charges on the originating object. Because the electrons are simply transferred from one object to the other, the total charge in both objects is conserved.

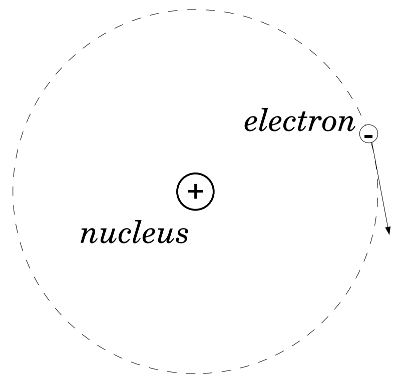


FIG. 1 Simple model of the atom, a negatively charged electron orbiting a positively charged nucleus.

The nucleus consists of protons and neutrons. Protons are positively charged with the exact opposite of the electron's charge. There are also other nuclear (NB, not 'nukular') forces, not treated here, perhaps towards the end.

B. Conductor vs Insulator

We distinguish between conductors and insulators. In conductors, charge can move freely, whereas in insulators, charge is localized.

In conductors, charge sits on the surface. It sits on the surface, because all charges repel each other, and will try to get as far away from each other as possible.

When two differently charged conductors are brought into contact, the electrostatic force will move the electrons from the negatively charged conductor to the positively charged conductor. While the charge is flowing, both object's charges are changing. The positively charged object is becoming less positively charged, and the negatively charged object is becoming less negatively charged. This flow of electrons will persist, until there is no more force acting on the charges, i.e. until both objects have the same charge. This charge flow is quantified by a current, $I = \frac{\Delta q}{\Delta t}$, i.e. the number of Coulombs per second, and the unit of current is A(mpere)=C/s.

C. Coulomb's law: force between charges

Charge is either positive or negative, like charges repel, opposite charges attract. The force on q_1 due to q_2, i.e. F_{12}, is drawn in the figure for several combinations of positive and negative charges. Note that F_{12} is always pointing in the opposite direction from F_{21}, a consequence of Newton's third law. When the charges are both positive (case a) or both negative (case c), the force is pointing

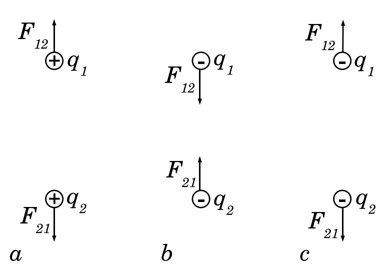


FIG. 2 Like charges repel, oppo-

away from the other charge, because the force is repulsive, and tries to move the charges away from each other. When the charges are opposite (case b), the force is pointing towards the other charge, because the interaction is attractive. The electrostatic (attractive of repulsive) force is a vector. The magnitude is given by Coulomb's law

$$F_{12} = k\frac{q_1 q_2}{r_{12}^2}$$

where $k = 8.99 \times 10^9$.

The full vector form of Coulomb's law is

$$\vec{F}_{12} = \pm k\frac{|Q_1 Q_2|}{r_{12}^2}\hat{r}_{12}$$

where the $\pm$ has to be chosen to represent attraction or repulsion. The factor $\hat{r}_{12}$ ensures that the force is pointing in the direction along the axis joining the charges.

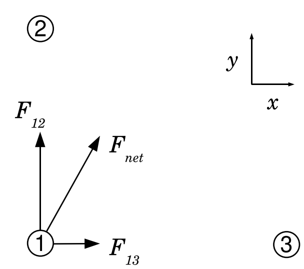
D. Interaction between multiple charges

The total (net) electrostatic force on an object (say 1) is the sum of all electrostatic forces acting on it (say from object 2, 3, ...). We can write

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \ldots$$

Example:

Two charges $Q_2 = 5.0\mu C$ and $Q_3 = 2.5\mu C$ are located at $(x, y) = (0, 0.50)$ and $(x, y) = (0.50, 0)$ respectively. $Q_1 = -1.0\mu C$. We calculate the net force acting on Q_1. $\vec{F}_{13}$ is attractive, so pointing towards Q_3



$$\vec{F}_{13} = k\frac{|Q_1 Q_3|}{r_{13}^2}\hat{r}_{13} = 9.0\times 10^{-9}\frac{1.0\mu C \times 2.5\mu C}{(0.50m)^2}\hat{x} = 0.090 \text{ N}\hat{x}$$

$\vec{F}_{12}$ is attractive, so pointing towards Q_2,

$$\vec{F}_{12} = k\frac{|Q_1 Q_2|}{r_{12}^2}\hat{r}_{12} = 9.0 \times 10^{-9}\frac{1.0\mu C \times 5.0\mu C}{(0.50m)^2}\hat{y} = 0.18 \text{ N}\hat{y}$$

The net force on Q_1 is

$$\vec{F}_{net} = 0.090 \text{ N}\hat{x} + 0.18 \text{ N}\hat{y} = \begin{pmatrix} 0.090 \\ 0.18 \end{pmatrix} \text{N}$$

E. The electric field

When several forces are acting on a charge q_0, then for each one of these forces, the force is proportional to q_0. Consider 3 charges Q_1, Q_2, Q_3, then the force on a fourth charge q_0 is equal to

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \vec{F}_{03}$$

and for all those terms, Coulomb's law holds, so all those terms are proportional to q_0. We can isolate that proportionality by dividing out the q_0 dependence, and rewrite

$$q_0\vec{E} = q_0\vec{E}_1 + q_0\vec{E}_2 + q_0\vec{E}_3$$

or, eliminating q_0,

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

We have now found the electrostatic field (or electric field), a vector quantity associated with charged objects. The electric field has unit N/C. The electric field indicates what direction the force on a positive test charge q_0 would be.

The superposition principle holds for electric fields as well.

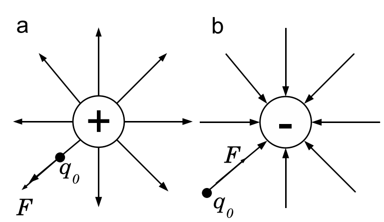
When drawing electrostatic fields, we often draw electrostatic field lines. These represent the force that would be acting on a positively charged 'test particle'. Electric field lines go from positive (or infinity if there is none) to negative charges (or infinity if there is none).

Generally, we can go from electric field to force and vice versa easily by $\vec{F} = q\vec{E}$.

Example:

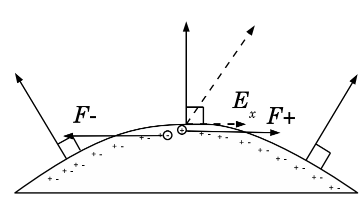
The electric field due to a spherically symmetric charge Q, or a point charge, can be found from Coulomb's law,

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{q_0}k\frac{Qq_0}{r^2}\hat{r} = k\frac{Q}{r^2}\hat{r}$$



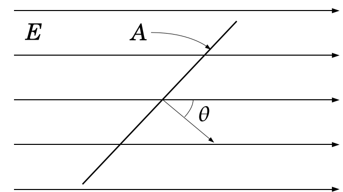
The electric field is pointing away from the conductor for a positive charge (case a in the figure), because the positive test charge q_0 would be repelled, whereas the electric field is pointing toward Q if $Q < 0$ (case b in the figure), because the positive test charge is attracted by Q.

Electrostatic field lines meet the surface of a conductor at 90°. In the figure, it can be seen that if the electric field would not be perpendicular to the surface, indicated by the dashed arrow, there would be component of the electric field E_x that points along the surface. This component would exert a force on the positive and negative charges indicated by F_+ and F_- respectively. This would separate the charges and the charges would keep moving until there is no more force component along the surface, i.e. when the electric field line is perpendicular to the surface.



F. Electric Flux and Gauss' law

The flux through a surface is given by $\Phi = EA\cos\theta$, where θ is the angle between the 'normal' of the surface and the field lines. Flux is largest when the angle is 0 or 180 degrees, 0 when 90 or 270. It has a sign which measures if the field lines are going in, or coming out. The stronger the electric field, the more field lines there are.



A gaussian surface is the surface of an imaginary object that is closed, i.e. without any holes in it.

For a point charge, the field is $E = kq/r^2$. The flux through the spherical gaussian surface A_1 at distance r_1 around the charge with surface area $4\pi r_1^2$ is

$$\Phi = EA_1 = k\frac{q}{r_1^2}4\pi r_1^2 = 4\pi kq = \frac{q}{\epsilon_0}$$

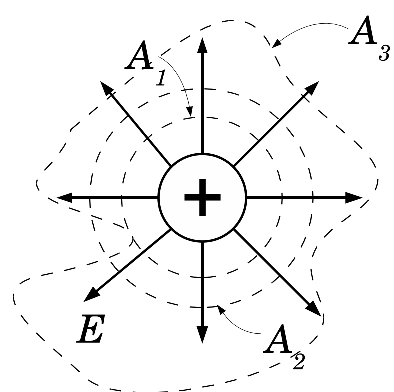
where in the last step we have defined $\epsilon = \frac{1}{4\pi k} = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{Nm}^2$, which is the permettivity of free space. The flux through the surface A_2 at distance of r_2 is equal to

$$\Phi = EA_2 = k\frac{q}{r_2^2}4\pi r_2^2 = 4\pi kq = \frac{q}{\epsilon_0}$$

It turns out that the flux is independent of the shape. Even the outcome for the gaussian surface A_3 (which will be hard to calculate, because the electric field varies along the surface) will be $\Phi = q/\varepsilon_0$

This is Gauss' law, it states that

$$\Phi = EA = \frac{q_{enc}}{\epsilon_0}$$



where q_{enc} is the enclosed charge. Gauss' law does several things

1. it measures the number of field lines that go through the gaussian surface

2. it measures the charge enclosed by the gaussian surface

3. it measures the field strength at the surface, adding all the components perpendicular to the gaussian surface

We can use Gauss' law to derive Coulomb's law, by reversing the calculation above. We strike a spherical gaussian surface with area $4\pi r^2$ around a point charge q_1, the electric field must be given by Gauss' law, and is $E = \Phi/A = q_1/\epsilon_0 4\pi r^2$, and the force on another charge q_2 is found by $F = q_2 E$, so we find

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$$

We can therefore treat electricity in terms of Gauss' law as well as Coulomb's law, they are equivalent.

We can also find the electric field between the plates of a parallel plate capacitor this way. Strike a cylindrical gaussian surface, going from inside one of the plates into the area between the plates. When the charge density on the plates is σ, the enclosed charge is $A\sigma$, where A is the surface area of one of the ends of the cylinder. No field lines go through the surface, except for the end which is in the space between the plates. There, the flux is EA, so Gauss' law leads to the expression for the electric field between the plates

$$E = \frac{\sigma}{\epsilon_0}$$

We can use Gauss' law to show that $E = 0$ within a charged conductor, because there will be no enclosed charge inside the material, because all the charge is on the surface.

G. Potential

The potential energy change by a force field $\vec{F}$ is opposite to the work done

$$\Delta U = -\Delta W$$

and because the work done is $\Delta W = \vec{F} \cdot \Delta\vec{r}$, we can write

$$\Delta U = -\vec{F} \cdot \Delta\vec{r}$$

In the context of electrostatics, the force is always equal to the charge under consideration q and the electric field $\vec{E}$ due to other charges around

$$\Delta U = -\vec{F} \cdot \Delta\vec{r} = -q\vec{E} \cdot \Delta\vec{r}$$

Because the $\Delta U \propto q$, we define a property *potential* V, units Volts, as

$$V = \frac{U}{q}$$

and in the context of electrostatics, this becomes

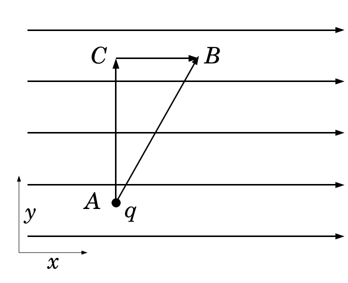
$$\Delta V = -\vec{E} \cdot \Delta\vec{r}$$

When the electric field varies along the path $\Delta\vec{r}$, we cannot readily evaluate this expression. We would have to split the path up in small pieces over which we assume the field is constant, and then use the methods of calculus to evaluate the potential. (We would be performing integration)

1. Constant electric fields

When the electric field is constant, we can simplify this greatly. A charge is moved from point A to B. The path $\Delta\vec{r}$ is specified by the change of the two coordinates Δx and Δy,

$$\vec{r} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$



so the potential can be expanded as

$$\Delta V = -\vec{E} \cdot \Delta\vec{r} = -\begin{pmatrix} E_x \\ E_y \end{pmatrix} \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -E_x\Delta x - E_y\Delta y$$

The electric field is pointing in the x direction, so the y component of the electric field is 0 ($E_y = 0$), and the potential becomes

$$\Delta V = -E_x\Delta x$$

The potential *decreases* in the direction of the electric field, because the electric field performs work, $\Delta W > 0$, so $\Delta V = -\Delta W/q < 0$.

Example: A charge is moved in the direction of an electric field with strength 10 N/C along a path of 1 m long. We calculate the change in potential

$$\Delta V = -E_x\Delta x = -10 \times 1 = -10 \text{ V}$$

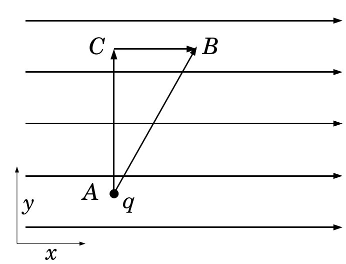
Note that the change in potential is independent of the charge.

Example: A charge is moved in a opposite direction of an electric field with strength 10 N/C along a path of 1 m long. We calculate the change in potential

$$\Delta V = -E_x\Delta x = -10 \times (-1) = 10 \text{ V}$$

Note that the change in potential is independent of the charge.

The electric field is a conservative field. The potential change is therefore independent on the path that we take. Going from point A to B in a straight line, we already calculated the potential change above. If we go through point C instead, the potential change is



$$\Delta V = \Delta V_{AC} + \Delta V_{CB} = -\underbrace{\vec{E} \cdot \hat{y}\Delta y}_{=0} - \vec{E} \cdot \hat{x}\Delta x$$

The first term is 0 because the path and the electric field are perpendicular. The second term evaluates to

$$\Delta V = -\vec{E} \cdot \hat{x}\Delta x = -E\Delta x$$

which is exactly equal to what we found above, as expected.

2. Conversion to kinetic energy

Because the sum of potential and kinetic energy is constant, the increase in potential energy must be opposite to the change in kinetic energy

$$K_E + U = \text{constant} \qquad \rightarrow \qquad \Delta K_E = -\Delta U$$

If the electric field is constant, we can write this as

$$\Delta K_E = -\Delta U = -q\Delta V$$

Example: A particle with 1.0 nC charge is initially at rest, and released at $t = 0$, in an electric field with strength of 10 N/C. We calculate the kinetic energy after the charge has moved 1 m.

$$\Delta K_E = -q\Delta V = qE_x\Delta x = 1.0 \times 10^{-9} \times 10 \times 1 = 10 \text{ nJ}$$

Although the potential is independent of charge, the change in kinetic energy is dependent on charge. If the mass of the particle is 20 g, we can calculate the final velocity from

$$\Delta K_E = \frac{1}{2}mv_f^2 - \underbrace{\frac{1}{2}mv_i^2}_{=0} = 10 \text{ nJ} \quad \rightarrow \quad v_f = \sqrt{\frac{2 \times 10 \times 10^{-9}}{2.0 \times 10^{-2}}} = 1.0 \times 10^{-3} \text{ m/s}$$

3. Potential vs. Kinematics

When the electric field is constant, we have a choice. We can solve the problem with kinematics as well as using potentials and arrive at the same answer.

When the electric field is not constant, we cannot use kinematics, because the force is changing along the path of motion. We also cannot use the expression $\Delta V = -E\Delta x$, because the electric field is changing as well. We would have to use methods of calculus to solve these problems. Physicists have solved these problems for you for several simple cases. It turns out that in these situations, we can still use our law that the change in kinetic energy is opposite to the change in potential energy, so

$$\Delta K_E = -q\Delta \tag{15}$$

and the potential has to be specified.

4. Varying electric fields and varying potentials

The electric field from a point charge or charged sphere is

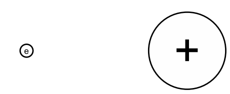
$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

where r is the distance to the center of the charge and the associated potential is

$$V = \frac{kQ}{r}$$

When r increases, we are moving away from the charge. The electric field is then doing work, the potential should therefore decrease as well. Close inspection of the equation for the potential shows us that it indeed decreases when moving away from the charge.

Example:
An electron is initially at rest, and is released 60 cm from the surface of a conducting sphere with charge 2.3 nC and diameter of 15 cm
The potential difference is the difference between the starting point (A) and the end point at the surface of the sphere (B). For a spherical charge distribution Q, the potential is a function of distance r from the center of the sphere is



$$V = \frac{kQ}{r}$$

so the potential difference is $\Delta V = V_B - V_A =$

$$V_B - V_A = \frac{kQ}{r_B} - \frac{kQ}{r_A} = 9.0 \times 10^9 \times 2.3 \times 10^{-9} \left(\frac{1}{0.075} - \frac{1}{0.075 + 0.60} \right) = 307 \text{ V}$$

The sum of kinetic energy and potential energy is constant,

$$K_A + U_A = K_B + U_B \quad , \quad K_B - K_A = U_A - U_B = q(V_A - V_B)$$

Now note that the charge of the electron is negative ($Q = -e$)

$$\frac{1}{2} m v_B^2 - \underbrace{\frac{1}{2} m v_A^2}_{=0} = -e(V_A - V_B) = e(V_B - V_A)$$

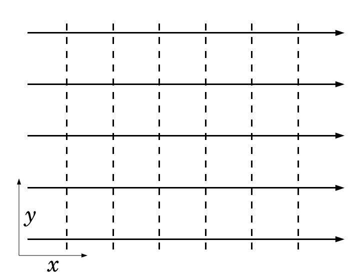
Solving for v_B, using the result from above for the potential difference

$$v_B = \sqrt{2e(V_B - V_A)/m} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 307}{9.1 \times 10^{-31}}} = 1.0 \times 10^7 \text{ m/s}$$

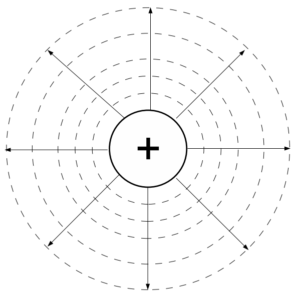
5. Equipotentials

A collection of points with the same potential is called an equipotential, for ‘equi’=same. Equipotentials intersect electric field lines always at 90°, because there is no work done along paths perpendicular to the force.

Equipotentials for a constant electric field pointing in the x direction are shown in the figure as dashed lines. They are perpendicular to field lines. When charges are released in this electric field, they will travel perpendicular to the equipotentials.



Equipotentials for a point charge or spherical charge distribution are spherical surfaces. The surfaces are indicated in the figure as dashed lines. The equipotential surfaces are again perpendicular to the electric field lines. If a charge is released in this electric field, it will move perpendicular to the equipotentials.

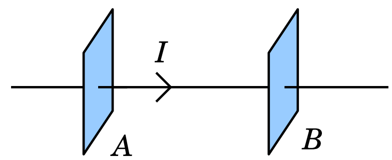


IV. DC CIRCUITS

DC circuits are circuits with a constant current. The DC stands for direct current, and should be compared to alternating current (AC) circuits, where the current changes over time. We will talk about AC circuits later in this course.

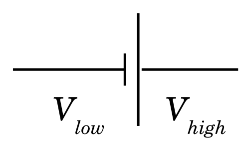
A. Wire

An ideal wire is schematically indicated by a thin line. The potential (voltage) is constant along the wire, independent of the current $I = \frac{\Delta q}{\Delta t}$. Charge is confined to the wire, i.e. two intersections at points A and B along the wire will have the same amount of charges flowing through them per unit time; the current at A and B are therefore the same, $I_A = I_B$.

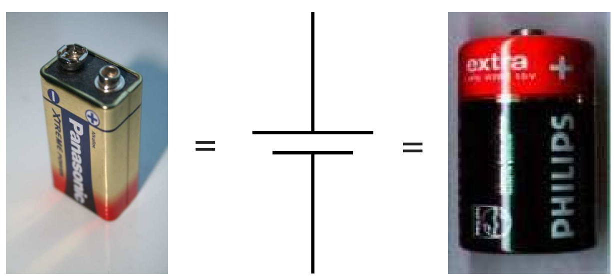


B. Voltage Source

A voltage source is schematically indicated by two lines perpendicular to a wire. The source has a potential difference $V = V_{high} - V_{low}$ between both sides. The terminal with the longer line is at higher potential than the shorter line. An ideal voltage source can maintain a constant voltage difference between the two terminals independent of the current.



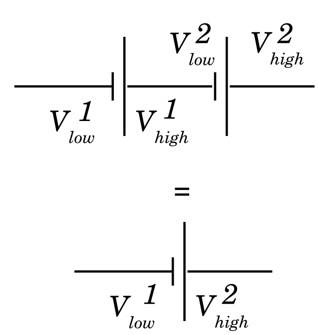
Example:



A 9V battery (picture on the left) has a potential difference of 9.0 V between the two terminals. A D-cell battery (picture on the right, common in heavy duty flash lights) has a potential difference of 1.5 V between the terminals. Car batteries have typically 12 V potential difference between the terminals.

When two sources are connected in series by a wire, the low-voltage terminal of source 2 is at the same potential as the high-voltage terminal of source 1, $V^2_{low} = V^1_{high}$, because the wire connects points with the same potential. Therefore

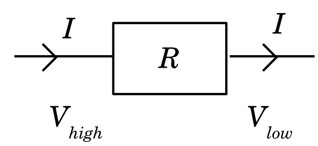
$$V_1 + V_2 = V^2_{high} \underbrace{-V^2_{low} + V^1_{high}}_{=0} - V_{low}1 = V^2_{high} - V^1_{low} = V_{12}$$



The total voltage $V_{12} = V_1 + V_2$. The two sources can therefore be replaced by a single source with a combined voltage of the sum of the two sources.

Example: In a maglite flashlight, two AAA batteries with each a voltage of 1.5 V between the terminals are connected in series, creating a combined voltage of 3.0 V which drives the light bulb in the flash light.

C. Resistor, Ohm's law



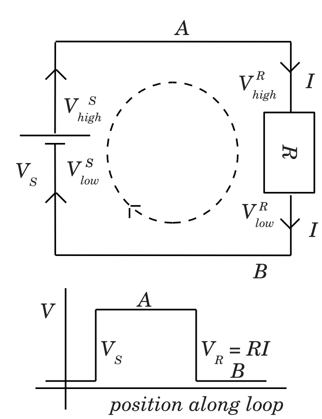
A resistor is schematically indicated in the figure. A cuurrent I flowing through the resistor causes a voltage difference between the input and output terminal of the resistor. The voltage at the input is high, and the voltage at the output is low. The voltage always decreases in the direction of the current, as opposed to a battery, where the voltage increases in the direction of the current. The voltage difference $V = V_{high} - V_{low}$ is proportional to the current I with proportionality R. The resistance R has units Ohm, or Ω. This is Ohm's law:

a current I flowing through a resistor R causes a voltage drop in the direction of the current equal to $V = RI$.

Example: A 1.0 mA current flowing through a resistor with resistance 10 kΩ. The voltage difference between the high and low voltage terminals is then equal to

$$V = RI = 10 \times 10^3 \times 1.0 \times 10^{-3} = 10 \text{ V}$$

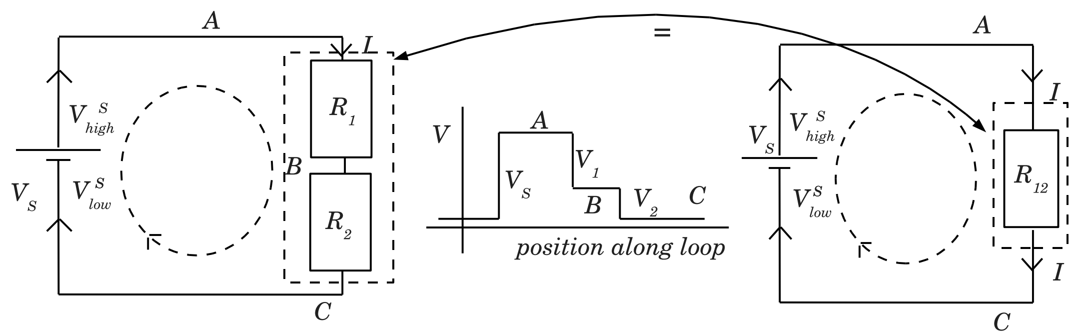
D. Basic circuit



A voltage source V_S is connected to a resistor R. The source is driving a current from the high-voltage terminal through the resistor to the low-voltage terminal.

We plot the potential along the indicated loop, following the current, starting at the low-voltage terminal at the battery. Going through the source, there is a potential increase of $V_S = V^S_{high} - V^S_{low}$. The following the current through the wire to point A and onward to the resistor, the potential does not change, because the potential is always constant along a wire. Then, going through the resistor, there is a potential decrease of $V_R = V^R_{high} - V^R_{low}$. Following the wire through point B back to the low-voltage terminal of the source, we arrive at our starting position. Because we are back at the same point, the potential must equal to the potential that we started at. Therefore, the potential increase at the battery V_S must be equal to the potential decrease V_R through the resistor $V_S = V_R$.

E. Resistors in series



Two resistors R_1 and R_2 are connected in series in the circuit. We plot the potential along the loop, starting at point A. Going through the source, the potential increases by V_S. There is no potential change along the top wire. Then there is a potential decrease of magnitude $V_1 = R_1 I_1$ going through R_1, and we are at point B. Another potential decrease of magnitude $V_2 = R_2 I_2$ occurs through resistors R_2. Then the potential is constant along the wire, until we arrive back at the low potential terminal of the battery in point A. We are now back at the starting potential again, therefore the potential increase at the battery V_S must be equal to the sum of the potential decreases at the resistors $V_S = V_1 + V_2$. Because all charges that flow out of R_1 must go into R_2 (there are no junctions into other branches that may carry current), the current flowing through both resistors must be the same, $I_1 = I_2 = I_{12}$. So

$$V_S = V_1 + V_2 = R_1 I_1 + R_2 I_2 = (R_1 + R_2) I_{12}$$

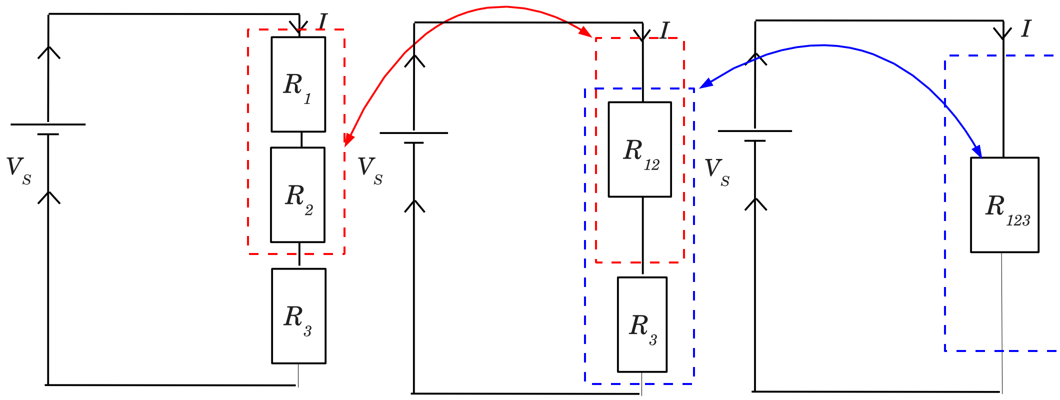
We then divide both sides by I_{12} and find Ohm's law for the equivalent circuit with a resistance of $R_{12} = R_1 + R_2$,

$$\frac{V_{12}}{I_{12}} = R_{12}$$

Summarizing, the equivalent resistance of a series configuration of two resistors R_1 and R_2 is $R_{12} = R_1 + R_2$, where the voltage across the equivalent resistor is $V_{12} = V_1 + V_2$, and the currents are equal $I_1 = I_2 = I_{12}$.

Example: A 10 V source is connected to a series configuration of two resistors $R_1 = 5\Omega$ and $R_2 = 10\Omega$. The equivalent resistance is equal to $R_{12} = R_1 + R_2 = 5 + 10 = 15\Omega$. The current is

$$I_{12} = \frac{V_{12}}{R_{12}} = \frac{V_S}{R_{12}} = \frac{10}{15} = 0.67A \quad (= I_1 = I_2)$$



We now expand this for three resistors R_1, R_2, R_3, following the same logic, using the equivalent resistance rule that we just found. We start with the circuit on the left, then substituting the equivalent resistance R_{12} for R_1 and R_2 (red substitution), using

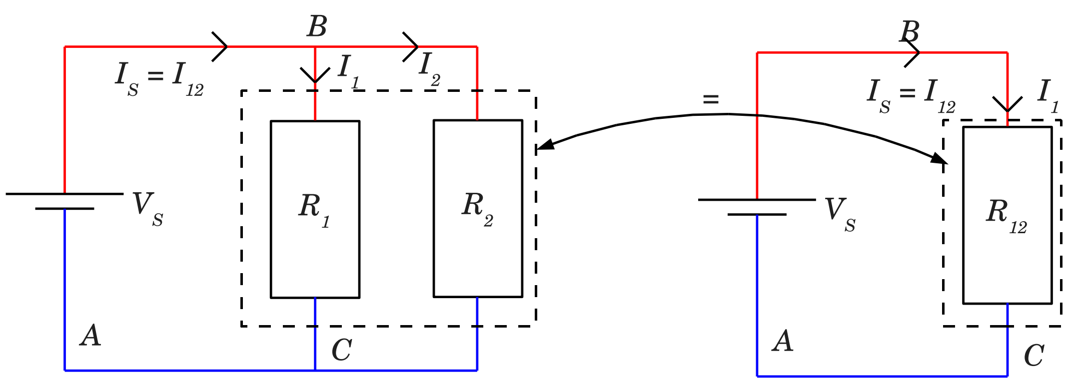
$$R_{12} = R_1 + R_2 \qquad I_{12} = I_1 = I_2 \qquad V_{12} = V_1 + V_2$$

Then we substitute (blue substitution) an equivalent resistance R_{123} for R_{12} and R_3, using

$$R_{123} = R_{12} + R_3 = R_1 + R_2 + R_3 \quad I_{123} = I_{12} = I_3 \quad V_{123} = V_{12} + V_3 = V_1 + V_2 + V_3$$

We see that the voltage across the equivalent resistance of the whole network is the sum of the voltages across all resistors, that all currents are equal, and that the equivalent resistance is the sum of all the resistances.

F. Parallel circuits



Tracing the potential in the direction of the current, starting at A, we first get a potential increase of V_S at the source, then a constant potential until we encounter the junction B. Here, the current splits into two separate currents I_1 and I_2, that recombine into a single current again at point C. Because the tops of the resistors (on the red side) are connected by an ideal wire, they are at the same potential. Similarly, the low potential sides at the bottom (blue wire) are at te same low potneital. The potential decrease V_1 across R_1 is therefore equal to the decrease V_2 across R_2 and equal to the potential difference between points B and C:

$$V_B - V_C = V_1 = V_2$$

Charges going into the junction B can go either towards R_1 or R_2, but nowhere else. Therefore, the total current going into the junction is equal to the total current coming out of the junction,

$$I_S = I_1 + I_2$$

Using Ohm's law for R_1 and R_2,

$$I_S = I_1 + I_2 = \frac{V_1}{R_1} + \frac{V_2}{R_2} = V_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

So, using $I_{12} = I_S$,

$$\frac{I_{12}}{V_{12}} = \frac{1}{R_{12}} = \frac{1}{R_1} + \frac{1}{R_2}$$

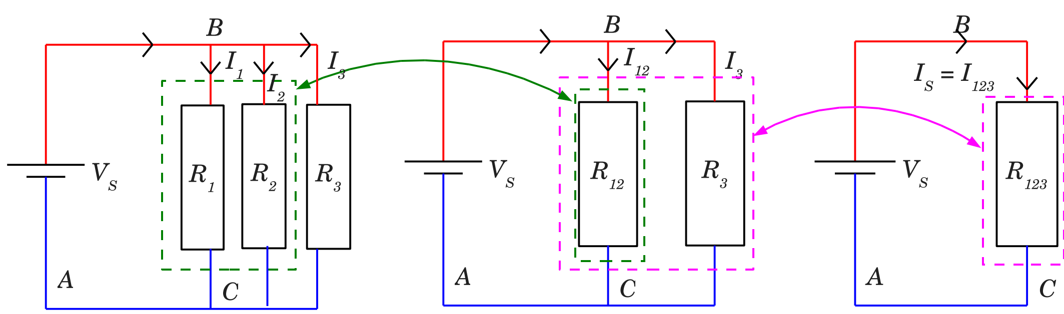
We can replace the parallel configuration of two resistors with a single resistor with resistance

$$R_{12} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Example: Two resistors $R_1 = 5\Omega$ and $R_2 = 10\Omega$ are in parallel, driven by a coltage source of 10 V. The equivalent resistance is $R_{12} = \left(\frac{1}{5}\frac{1}{10}\right) = 3.3\Omega$, and the current is $I_{12} = \frac{V_{12}}{R_{12}} = 3A$. Notice that, although the values are exactly the same as in the previous example, the outcome is very different, because the resistors are now arranged in parallel rather than series.

Summarizing, for two parallel resistors, we find

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_{12}} \qquad I_1 + I_2 = I_{12} \qquad V_1 = V_2 = V_{12}$$



Going to a situation of three parallel resistors, similar reasoning, using the rules that we have just derived, we first substitute R_{12} for R_1 and R_2 (green substitution)

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_{12}} \qquad I_1 + I_2 = I_{12} \qquad V_1 = V_2 = V_{12}$$

and then substitute R_{123} for R_{12} and R_3 (magenta substitution)

$$\frac{1}{R_{13}} + \frac{1}{R_3} = \frac{1}{R_{123}} \qquad I_{12} + I_3 = I_{123} \qquad V_{12} = V_3 = V_{123}$$

combining both results

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}=\frac{1}{R_{123}} \qquad I_1+I_2+I_3=I_{123} \qquad V_1=V_2=V_3=V_{123}$$

Generalizing, for a series of n parallel resistors, the equivalent resistance is

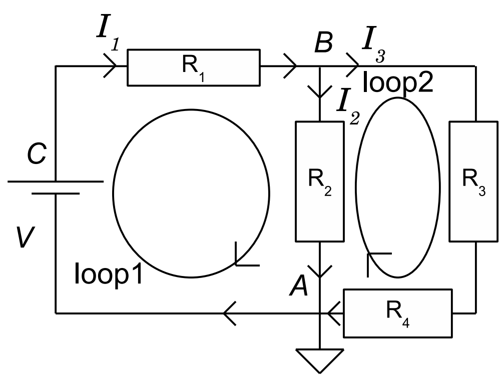
$$\frac{1}{R_{1\ldots n}}=\sum_{i=1}^{n}\frac{1}{R_i} \qquad ,$$

the current is the sum of all currents

$$I_{1\ldots n}=\sum_{i=1}^{n} I_i \qquad ,$$

and the voltages are all the same.

G. Combinations of parallel and series circuits



The circuit drawn is connected to a battery supplying 7.5 V. The resistors are $R_1 = 10\Omega$, $R_2 = 15\Omega$, $R_3 = 5\Omega$, $R_4 = 15\Omega$. The point A is grounded.

Remember the mantra: Follow the current! We follow the current coming from the battery. The current I_1 runs through R_1, then splits up at point B. Two branches continue, one branch carrying the current I_2 with R_2 in it, and one branch carrying the current I_3 with R_3 and R_4 in it. The currents then recombine at A, after which we reach the low-voltage terminal of the battery. See the indicated currents in the graph. We divide the network up in parallel and series pieces, substituting equivalent resistances along the way.

We start at the highest level, and work our way in. Ultimately, we can substitute an equivalent resistor R_{1234} for the whole circuit. Zooming in one level, we see that the current that runs through R_1 splits up in B, and recombines in A. The voltage drop between the points A and B must add to the voltage drop across R_1, so R_1 is in series with the equivalent curcuit between points A and B, which is R_{234},

$$R_{1234}=R_1+R_{234} \qquad I_{1234}=I_1=I_{234} \qquad V_1+V_{234}=V_{1234} \tag{16}$$

The current running through R_1 is equal to the current running through R_{234} and equal to the current running through the whole circuit.

Zoom in one step on the network with 2,3,4, we see that the current splits up in point B and recombines in point A. The left branch with R_2 is in parallel with the right branch with resistors R_3 and R_4 in it, so

$$\frac{1}{R_{234}} = \frac{1}{R_2} + \frac{1}{R_{34}} \qquad V_2 = V_{34} = V_{234} \qquad I_2 + I_{34} = I_{234} \tag{17}$$

Finally, the branch with R_3 and R_4 is a series network

$$R_{34} = R_3 + R_4 \qquad I_3 = I_4 = I_{34} \tag{18}$$

Now putting in the numbers, we get

$$\begin{aligned} R_{34} &= R_3 + R_4 = 20\Omega \\ R_{234} &= 1/\left(\tfrac{1}{R_2} + \tfrac{1}{R_{34}}\right) = 8.57\Omega \\ R_{1234} &= R_1 + R_{234} = 18.57\Omega \end{aligned}$$

Starting from the highest level again, we know that $I_{1234} = I_1 = I_{234}$ because that is a series configuration, and that is given by Ohm's law,

$$I_{1234} = \frac{V_{1234}}{R_{1234}} = \frac{7.5\text{ V}}{18.57\Omega} = \underline{0.404\text{ A} = I_1}$$

and it holds that

$$\underline{V_1 = R_1 I_1 = 4.0\text{ V}}$$

Following loop1, we know from step (16) that $V_1 + V_{234} = V_{1234}$, so we can V_1 from 7.5 V to get the voltage between points A and B,

$$V_{234} = 7.5 - 4.0 = 3.5\text{ V}$$

From part (17), we know $\underline{V_2} = V_{34} = V_{234} = \underline{3.5\text{ V}}$. We can now apply Ohm's law for R_2

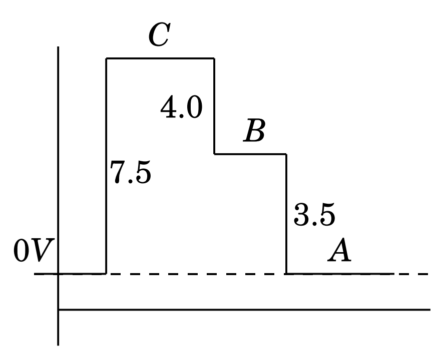
$$\underline{I_2} = \frac{V_2}{R_2} = \frac{3.5\text{ V}}{15\Omega} = \underline{0.23\text{ A}}$$

From part (18) we know $I_3 = I_4 = I_{34}$, which in turn we find from Ohm's law

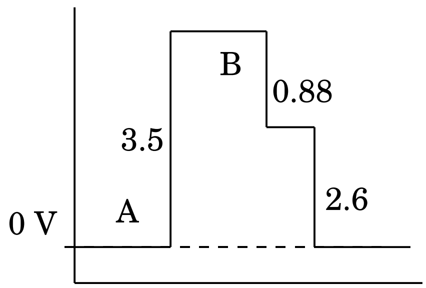
$$I_{34} = \frac{V_{34}}{R_{34}} = 3.5/20 = 0.175\text{ A}$$

so $\underline{I_3 = I_4 = I_{34} = 0.175\text{ A}}$. Finally we find

$$\underline{V_3 = R_3 I_3 = 0.88\text{ V}} \qquad \underline{V_4 = R_4 I_4 = 2.62\text{ V}}$$



We start out at the low voltage terminal of the battery and move in the direction of the current. Going through the battery, we experience a potential increase of 7.5 V and get to point C. Then we move with the current through R_1, so we experience a decrease in voltage of magnitude V_1 and arrive at point B. Then we go from point B to point A and experience a voltage decrease that is equal to $V_2 = V_{34}$ and arrive at point A. We are then back at the low voltage terminal of the battery. The potential at point A is 0 V, so we mark the graph accordingly.



We start at point A, going upwards through R_2 *against* the direction of the current, so we experience a voltage *increase* of V_2 and arrive at point B. We then follow I_3 through R_3 where we get a potential decrease, because we are now walking in the direction of the current, and another decrease when we go through R_4 and we are back at point A. This is ground, so we adjust the scale accordingly.

V. MAGNETISM

A magnetic field is given by field lines $\vec{B}$, which go from North to South poles. Like poles repel, and opposite poles attract.

The force of a magnetic field $\vec{B}$ on a moving charge q with velocity $\vec{v}$ is given by Lorentz' law

$$\vec{F} = q\vec{v} \times \vec{B}$$

The cross product between two vectors produces a vector perpendicular to both, and a magnitude given by $|\vec{a} \times \vec{b}| = ab \sin\alpha$, where α is the enclosed angle. Note the difference between this "cross", or "outer" product, and the "dot" or "inner" product.

We use the right hand rule to evaluate the direction. Point your fingers in the $q\vec{v}$ direction, curve your fingers towards the magnetic field $\vec{B}$, and your thumb indicates the direction of the force $\vec{F}$.

Note that reversing the sign of the charge reverses the direction of the force. Therefore, opposite charges flowing in the same direction through a magnetic field will get separated. This can happen with an ionic solvent, or with the charge carriers in a conductor. The latter effect is called the "Hall" effect.

The force on a current carrying wire with length L can be found with a simple modification. The product $q\vec{v}$ needs to be replaced by $\vec{I}L$, and we arrive at

$$\vec{F} = \vec{I} \times \vec{B}$$

A charged particle with mass m and charge q describes a circular orbit in a magnetic field B. We use the balance between the centripetal force $F_c = mv^2/r$ and the Lorentz force $F = qvB$ to find

$$\frac{mv^2}{r} = qvB \quad \rightarrow \quad r = \frac{mv}{qB}$$

If we know the velocity and charge of a particle, we can use this experiment to find it's mass.

We can also use the left-hand rule to find the direction of the force. In that case, point your fingers in the direction of $q\vec{v}$, 'catch' the magnetic field lines in the palm of your hand, and the thumb indicates the direction of the force. I find this rule much more clear than the 'right-hand rule', but note that they will give you the same answer.

The magnetic field lines point in the direction that the North pole of a compass would point. This means that the North pole of the earth is actually the magnetic South pole.

A rectangular loop carrying a current in a magnetic field produces a torque. The torque is given by $\vec{\tau} = \vec{F} \times \vec{d}$. Of the four sides of the loop, the sides parallel to the magnetic field do not produce force. The sides perpendicular produce a force, and they will be pointing in opposite directions, because the current has opposite direction. The torque is then given by $\tau = ABI$, where A is the surface area of the loop. This is independent of the shape of the loop. When the loop starts rotating, the angle between arm and force changes, so the torque decreases. The final expression of torque is thus $\tau = ABI \sin\alpha$, where α is the angle between the force and arm, or equivalently, the angle between the magnetic field and the vector perpendicular to the plane of the loop. The torque is proportional to the number of loops.

A current produces a magnetic field, and is given by Ampère's law

$$B = \frac{\mu_0 I}{2\pi r}$$

where $\mu_0 = 4\pi \times 10^{-7} Tm/A$ is the magnetic permeability of free space (vacuum/sufficiently clean air).

The direction of the field lines by a magnetic field is given by a modified right hand rule. Point your right-hand thumb in the direction of the current, and your finger will point in the direction of the magenetic field. The field lines turn around the wire in loops.

These fiels add, and the field exactly between two wires carrying the same current is 0 or $2B$ if the currents have the same or opposite direction respectively.

VI. MAGNETIC INDUCTION

We define magnetic flux in a similar manner as the electric flux, as the number of field lines threading a surface, symbolically

$$\Phi_B = BA\cos\alpha$$

where α is the angle between the normal of the surface and the magnetic field lines. So the flux is maximum when the field lines are perpendicular to the surface, and 0 when parallel. The unit of flux is Wb (Weber), or Tm^2.

When a magnet is moving towards a coil, it induces a voltage across the end terminals of the coil. Note that the coil is not closed at this point, so no current can flow. We get to this situation below. This voltage is due to a change in the flux threading the coil. Faraday's law of induction states that this *voltage* V is related to the change in flux by

$$V = N\frac{\Delta\Phi}{\Delta t}$$

here N is the number of windings in the coil. We can create a time-changing flux in several manners. We can move the magnet towards the coil, we can move the coil towards the magnet. We can also spin the magnet in front of the coil, and we can also sping the coil through the magnetic field.

Now we close the loop, such that a current can flow. Having a changing flux still creates a voltage, but now a current can flow. Note that at this point, we cannot calculate the magnitude of the current, because we don't know how to calculate the 'resistance' of the coil (it is *not* simply $R = \rho L/A$). The current flowing through the coil will create a magnetic field, the direction of which we can find using the right-hand rule: thumb pointing in the direction of the current, fingers curve in the direction of the magnetic field. We can treat this magnetic field as if it was due to a magnet, placed at the center of the coil, in a direction perpendicular to the plane of the coil, i.e. parallel to the normal of the coil. Now Lenz's law states that this induced magnetic field will be oriented such that it opposes the motion of the magnet. So when the North pole of the magnet is moving towards a coil, the induced current will cause a North pole to form on the side of the coil closest to the approaching magnet. We know that North poles repel, so we have to perform work to oppose this force. Physically, the fact that we have to do work is because energy is generated in the coil, and this has to come from somewhere. Now if we were to move the magnet away from the coil, the coil will start to attract the magnet, a South pole will form on the front of the coil.

VII. AC CIRCUITS

AC (Alternating Current) circuits are circuits with a current that changes direction at a frequency f, i.e. f times per second. It should be compared to DC (Direct Current) circuits, the type of circuit we have discussed until now, which has a constant direction of the current.

The voltage source driving such a circuit has a time dependence

$$V(t) = V_{max} \sin(\omega t) \quad ,$$

where $\omega = 2\pi f t = 2\pi t/T$ is the angular frequency that we encountered before, units rad/s. T is the period of a full cycle, unit s(econd). The voltage alternates in a sinusoidal manner between V_{max} and $-V_{max}$.

We draw this time dependence as a "phasor", a vector in a x, y-plot, that is rotating in counterclockwise direction. The x and y components of this phasor are given by

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = V_{max} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} \quad ,$$

so the voltage can be found by looking at the y-component of the phasor, i.e. the projection of the phasor on the y-axis.

Ohm's law for an AC circuit with only a resistor R holds

$$R = \frac{V}{I} \quad ,$$

but now V and I are both a function of time, so we write them as $V(t)$ and $I(t)$

$$R = \frac{V(t)}{I(t)} \quad .$$

We can also draw the current as a phasor, and it would always have the same angle as the voltage, and we describe this situation as "voltage and current are in phase".

We calculate the RMS value of a time-varying signal by taking the square first, then the mean, and finally the (square) root. The acronym can be read as "Root of Mean of Squared", or Root(Mean(Squared(signal))). The square of the time-varying signal is

$$V^2(t) = V_{max}^2 \left(\sin(\omega t)\right)^2$$

This describes an alternating signal, always positive, going from 0 to V_{max}^2. The mean (=average) value of this is $\frac{1}{2}V_{max}^2$. Finally taking the square root, we find

$$V_{RMS} = \frac{1}{\sqrt{2}} V_{max}$$

We can always define an RMS value for a time-varying signal, even if the signal does not describe a sine wave, like when it is triangular, a square wave, or a sawtooth shape. The equation linking V_{RMS} and V_{max} would be a little different, but that is beyond the scope of this class.

The RMS value occurs naturally when we discuss power dissipated in an AC circuit. Consider a simple circuit with only a resistor, the power dissipated becomes a function of time, but we can still use our previous expression of power $P = RI^2$, except we make the time dependence explicit,

$$P(t) = RI^2(t) = RI_{max}^2 \sin^2 \omega t \quad .$$

This is always positive, varying from 0 to RI_{max}^2. To calculate the average power dissipated in the resistor, we take the mean value of these two extremes, just like we did above, and find for the average power P_{av}

$$P_{av} = \frac{1}{2} RI_{max}^2 = RI_{RMS}^2$$

The RMS value appears automatically. We can also write P_{av} in terms of the RMS voltage

$$P_{av} = \frac{V_{RMS}^2}{R}$$

The current-voltage relation for a capacitor is more complicated. We have to combine two equations that we have used before, making the time dependence explicit

$$C = \frac{Q(t)}{V(t)} \quad \text{and} \quad I(t) = \frac{\Delta Q(t)}{\Delta t} \quad .$$

We substitute the first one in the second one

$$I(t) = \frac{\Delta(CV(t))}{\Delta t} = C \frac{\Delta V(t)}{\Delta t}$$

When $V(t) = V_{max} \sin \omega t$, it can be shown that $\frac{\Delta V(t)}{\Delta t} = \omega V_{max} \cos \omega t$. You can verify this by drawing the sine wave for $V(t)$, and reason that you'll 1) get a cosine for $\Delta V/\Delta t$, 2) keep the factor V_{max}, 3) pick up a factor ω, because a higher frequency will mean a compression of the time scale, and therefore a larger change in ΔV in the time interval Δt. If you know calculus, you could calculate this directly by taking the derivative of the voltage with respect to time, $\frac{\mathrm{d}V}{\mathrm{d}t}$, and arrive at the same result. Substituting this expression for $\Delta V/\Delta t$ in the equation for $I(t)$ above, we find

$$I(t) = \omega C V_{max} \cos(\omega t)$$

and the maximum current

$$I_{max} = \omega C V_{max} = V_{max}/X_C$$

where in the last step we have introduced the "admittance" of the capacitance, $X_C = 1/\omega C$, units of Ω. Note that although we can write this as

$$X_C = \frac{V_{max}}{I_{max}} = \frac{V_{RMS}}{I_{RMS}}$$

and it therefore has the same form as Ohm's law, even the same unit, X_C is not the same as R, because current and voltage are out of phase. We would need a modification to Ohm's law to account for this, which we will get to later.

Because the current varies as $\cos \omega t$ while the voltage varies as $\sin \omega t$, the angle between the phasors is 90 degrees, and we say they are "out of phase".