The Average Speed on the Highway

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The following problem has appeared in elementary statistics books: You are driving on the highway and adjust your speed until the number of cars that you pass is equal to the number of cars that pass you. Is your speed the median speed or the mean speed of the cars on the highway?

Students are expected to answer that since the number of cars going slower than you are is equal to the number of cars going faster than you are, your speed is the median speed. This certainly is true of the cars that you *see*, but that isn't what the problem asks, and it isn't the correct answer. The surprising results that we will derive illustrate the importance of careful and precise thinking when sampling.

Here's an example. Imagine a highway with five lanes. In one lane, cars travel 20 miles per hour with two cars per mile. The other lanes have cars traveling 45 miles per hour with seven cars per mile, 55 miles per hour with two cars per mile, 65 miles per hour with three cars per mile, and 70 miles per hour with six cars per mile. The cars' median speed is 55 miles per hour since there are nine cars per mile going faster than 55 miles per hour and nine cars per mile going slower.

When driving you cannot directly observe the number of cars moving at each speed. What will you see if you drive at the median speed of 55 mph? The number of cars going, say, 45 mph that you pass in an hour is equal to $(55 - 45) \cdot 7 = 70$. The computations for all of the speeds are given below:

Mph	Number of cars per mile	Number you pass per hour	Number that pass you per hour
20	2	70	0
45	7	70	0
55	2	0	0
65	3	0	30
70	6	0	90
Total	20	140	120

The table shows that if you travel at the median speed, then the number of cars that you pass is greater than the number that pass you.

How *can* you determine the median speed of traffic using only what you see from your car? Suppose you are traveling at s miles per hour. If a lane of cars is going x miles per hour, with d_x cars per mile and $x < s$, then you pass $(s - x)d_x$ cars in the next hour. If $x > s$, then $(x - s)d_x$ cars pass you in the next hour.

Let N_x be the number of cars traveling at speed x that you pass or that pass you in the next hour. Since $N_x = |x - s|d_x$, you can determine the number of cars going at speed x by $d_x = \frac{N_x}{|x-s|}$. It follows that your speed s is the median speed if

$$\sum_{x<s} \frac{N_x}{|x - s|} = \sum_{x>s} \frac{N_x}{|x - s|}.$$

For instance, in the example, if $s = 55$ we have

$$\frac{70}{35} + \frac{70}{10} = \frac{30}{10} + \frac{90}{15},$$

so 55 mph is the median speed. Medians can usually be found just by counting, but if you are driving it is impossible to find the median speed of the cars on the highway unless you can accurately determine the actual speed of each car you see.

By a similar line of reasoning, it is not hard to see that your percentile rank in the distribution of speeds on the highway is *not* given simply by the proportion of cars you pass out of the total number that you see,

$$\frac{\sum_{x<s} N_x}{\sum_x N_x},$$

but rather by the *weighted* proportion

$$\frac{\sum_{x<s} \frac{N_x}{|x-s|}}{\sum_x \frac{N_x}{|x-s|}}.$$

Now what about the mean speed? In our example the mean speed of cars is

$$\frac{2 \cdot 20 + 7 \cdot 45 + 2 \cdot 55 + 3 \cdot 65 + 6 \cdot 70}{2 + 7 + 2 + 3 + 6} = 54 \text{ mph}.$$

Surprisingly, then, if you are traveling at the mean speed then the number of cars that you pass is equal to the number that pass you:

Mph	Number of cars per mile	Number you pass per hour	Number that pass you per hour
20	2	68	0
45	7	63	0
55	2	0	2
65	3	0	33
70	6	0	96
Total	20	131	131

This result is true in general. If you adjust your speed so that as many cars pass you as you pass, then your speed is the mean speed of all the other cars on the highway. That is, the correct answer to the question at the beginning of this article is the mean! To see this, suppose you are traveling at a constant speed s for one hour so that the number of cars that you pass is equal to the number of cars that pass you. Cars going a given speed x are spaced d_x cars per mile. The number of cars that you pass is

$$\sum_{x\leq s}(s-x)d_x = s\sum_{x\leq s} d_x - \sum_{x\leq s} x\cdot d_x.$$

The number that pass you is

$$\sum_{x>s}(x-s)d_x = \sum_{x>s} x\cdot d_x - s\sum_{x>s} d_x.$$

If these are equal, then

$$s\sum_{x\leq s} d_x - \sum_{x\leq s} x\cdot d_x = \sum_{x>s} x\cdot d_x - s\sum_{x>s} d_x,$$

so

$$s = \frac{\sum x\cdot d_x}{\sum d_x} = \frac{\text{sum of speeds of all cars per mile}}{\text{number of cars per mile}} = \text{mean speed of all cars.}$$

The mean speed, then, isn't hard to determine—just adjust your speed to be the median speed of the cars you see!

Mathematics in Fiction

Charlie Marion (Lakeland High School, Shrub Oak, New York) noticed the following passage in *Death Beam*, by Robert Moss (Berkley Books, 1982), p. 269:

> We've perfected a continuous-wave chemical laser. We use thermonuclear combustion to produce atomic fluorine that is expanded through a set of supersonic nozzles and mixed in with deuterium when it flows into an optical cavity. There is a device like an inverted telescope at the aperture through which the laser exits that expands it into a wide beam capable of traveling huge distances. Our prototype laser guns have a power of ten megawatts, an aperture twelve meters wide, and a jitter of about a hundred nanoradians. In other words, we think we can aim them with an error of only four feet over a distance of 7,575 miles.

The author's engineering may be nonsense but his trigonometry is right on. Mr. Marion writes, "Whoa! Let's take a closer look at the math in those last two sentences. Let's see, What *is* (7575)(5280) tan(100\1,000,000,000 radians)? 3.9996 feet—close enough!