Chance
Encounters

"Two months after introducing a new game, the Crazy Al Casino is broke and is forced to go out of business. How can this game favor the player even though its component parts favor the casino?"

Pondering Parrondo's Paradox

Mark Schilling
California State University at Northridge

My colleagues and I sometimes enjoy reminiscing about the silly old TV commercials we saw as kids, like the one where a used car dealer always had a large exotic animal (a cheetah, an alligator, ...) on a leash while he hawked his latest clunkers. One recollection (that admittedly may be apocryphal) was of a man called "Crazy Al," who liked to shout "WE LOSE MONEY ON EVERY STEREO WE SELL! HOW DO WE DO IT?? VOLUME!!"

That a system resulting in losses could somehow lead to an overall gain seems absurd. Yet physicist Juan Parrondo has recently devised a type of gambling game that actually combines two losing strategies together to make—remarkably—a winning one. The following tale presents an example of this paradoxical kind of game.

The management of the Crazy Al Casino has decided to introduce a new game of chance for its customers. The equipment for this game is a European roulette wheel. This wheel contains 37 slots numbered from 0 to 36 in which a ball can fall when the wheel is spun. (In American roulette there is also a 00 slot.) Each number is equally likely. Half of the numbers from 1 to 36 are red, and the other half are black; the number 0 is colored green.

What is unusual about this game is that the bettor's chance of winning depends on his or her capital at the time the wheel is spun, where we define capital to be the total number of chips possessed. (We assume that all chips have equal value.) Specifically, if the player's capital is a multiple of three, she wins a chip if the number that turns up is a 1, 2, or 3 with resulting win probability 3/37; otherwise she wins if the outcome is any number between 1 and 28 (P(Win) = 28/37).

Naively, gamblers are likely to find this game attractive, since there is a very high chance of winning in two of the three possible cases. If one can assume that the player's capital modulo 3 will be equal to 0, 1, or 2 with equal frequency, then her average chance of winning would be 1/3 (3/37) + 2/3 (28/37) = 59/111 = .532 > 1/2, so the game seems to favor the player.

In reality this game favors the casino, because it turns out that in repeated plays the gambler's capital will be divisible by three, much more than a third of the time. To find the long run frequency of this occurrence, we can model the player's capital (mod 3) as a *Markov chain* with *states* 0, 1 and 2. A Markov chain is a sequence of states in which each state depends only upon chance and the previous state. The *transition matrix* $\boldsymbol{M}$ of this Markov chain is shown below. The (i, j) entry represents the chance that, when the chain is in state i, it will go next to state j:

$$\boldsymbol{M} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 0 & \frac{3}{37} & \frac{34}{37} \\ \frac{9}{37} & 0 & \frac{28}{37} \\ \frac{28}{37} & \frac{9}{37} & 0 \end{bmatrix} \end{array}.$$

Standard Markov chain theory says that the long run behavior of the chain can be determined by multiplying M by itself a large number of times; in the limit we obtain the *stationary distribution* of the chain in each row:

$$\lim_{n\to\infty} \boldsymbol{M}^n = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} .389 & .145 & .466 \\ .389 & .145 & .466 \\ .389 & .145 & .466 \end{bmatrix} \end{array}.$$

The entry in each column is the limiting probability that the chain is in that state. Thus when a bettor plays the game many times, his or her capital will actually be a multiple of three about 38.9% of the time, so the bettor's average chance of winning is .389(3/37) + (.145+.466) (28/37) = .494 < 1/2. Thus the game favors the casino in the long run.

Not content to quit while ahead, however, the Crazy Al Casino comes up with an even fancier idea which they call

"Beat the Wheel." Bettors can combine play of the previous game, which we shall now call Game A, with another game, Game B. Game B is a standard roulette gamble: the player bets on either red or black and wins or loses a chip depending on whether the color she bets on comes up. Game B also favors the casino, as the chance that the player wins is 18/37, which is less than 1/2.

Of course if the player can switch between Games A and B at will, she will want to play Game B whenever her capital is a multiple of three and Game A otherwise, and it is easy to see that the casino will rapidly lose money if this is allowed. Thus the casino forbids this, offering instead several allowable ways to play "Beat the Wheel." For instance, the bettor can alternate between the games (either ABABA... or AABBAABBAA... is allowed). In another variation, the croupier tosses a fair coin to determine which game the players play each time—i.e., the sequence of plays of Games A and B is completely random.

Two months after introducing "Beat the Wheel," the Crazy Al Casino is broke and is forced to go out of business. How did it happen? Were the bettors who played "Beat the Wheel" just unusually lucky? Or does "Beat the Wheel" somehow favor the player even though its two component games favor the casino?

In fact "Beat the Wheel" *does* favor the player! This is an instance of Parrondo's paradox. We have combined two losing strategies to form a winning strategy. Let's now investigate how this can happen. We can again model the bettor's situation as a Markov chain. Now there are six states: 0A, 1A, 2A, 0B, 1B and 2B, where the number indicates the bettor's capital (mod 3) and the letter represents the game that will be played next. This time the transition matrix is a 6×6 array. In each version of "Beat the Wheel" the bettor's capital will be a multiple of three (the case in which the casino is likely to win) much closer to a third of the time than in Game A alone.

For instance, when Games A and B are played in random order, calculating the stationary distribution of the chain reveals that the 0 state occurs only 34.6% of the time. The bettor's average win probability in this game is therefore $.346(3/37) + (1-.346)(28/37) = .505$, which means that the bettor will win a bit more than half of the time.

What causes Parrondo's paradox? Note that in our example the two component games are not independent of each other—if they were, the compound game would necessarily be a losing game for the bettor as well. Because of their dependency, the intermittent switching to Game B serves to equalize the chances of the bettor's capital being 0, 1 or 2 (mod 3). This reduces the chances that the bettor is in an unfavorable position (capital congruent to 0 mod 3) when Game A is played.

Here is a second illustration (patterned loosely after one by Philips and Feldman) that may help to demystify Parrondo's paradox. Consider the following two games: In Game 1 both the player's capital and the toss of a fair coin together determine the payoff. In Game 2 there is no random mechanism. The payoff structure of these games is shown below:

Game 1

	Payoffs	
If capital is:	Heads	Tails
Odd	51	−1
Even	1	−101

Game 2

If capital is:	Payoffs
Odd	−2
Even	−1

When only Game 1 is played, the bettor's capital alternates between even and odd, and the game is clearly unfavorable to the bettor since the large potential loss in the "Even" row outweighs the potential gain in the "Odd" row. Game 2 is obviously unfavorable also. But when a mixture of the two games is played, the combination can favor the gambler. This is because with Game 2 the player's resulting capital is always odd. Thus if the games are alternated starting with Game 2, then all plays of Game 1 will favor the bettor, and the gains from Game 1 will more than compensate for the losses from playing Game 2.

Whether Parrondo's paradox becomes known merely as another in the long list of probabilistic paradoxes remains to be seen. Parrondo's game is actually a discrete-time version of a mathematical process known as a *Brownian ratchet*. Physicists are interested in Brownian ratchets because it might be possible to devise molecular versions that can utilize random thermal fluctuations to obtain directed movement. Other areas of possible application are economics (including investment strategies), social networks and evolutionary biology. ■

Further Reading

G.P. Harmer and D. Abbott, Losing strategies can win by Parrondo's paradox, *Nature* (London), Vol. 402, No. 6764 (1999), 864.

T.K. Philips and A. B. Feldman, Parrondo's Paradox is not Paradoxical, *Social Science Research Network*, (August 2004). Available at `ssm.com/abstract=581521`.

M. F. Schilling, A Switch in Time Pays Fine, *Math Horizons*, Vol. 11, No. 1 (September 2003), 21–22.