Probability: From Monte Carlo to Geometry

The interplay between probability and other areas of mathematics often occurs in unexpected places. Below you will see how probability can be applied to problems in calculus, the estimation of e, and even to geometrical proofs.

To begin, let's look at a class of procedures known collectively as the Monte Carlo method, an approach which has become increasingly popular in recent years due to advances in computer technology. Named after the famous casino in Monaco, these procedures involve the use of a computer to generate a large number of random values which simulate real world or mathematical phenomena. The Monte Carlo method is effective because (1) modern high speed computers can create an extremely large amount of random data very quickly, and (2) the patterns produced by random observations, though haphazard in the short run, are quite predictable and well behaved in the long run.

Monte Carlo Integration

Suppose we have a definite integral such as

$$\int_0^2 \left(x^3+1\right)^{-1/2} dx,$$

which cannot be evaluated by standard analytic methods since $\left(x^3+1\right)^{-1/2}$ has no elementary anti-derivative. The approximate value of the integral can be obtained by numerical integration or by using a power series expansion of the integrand. Another way, however, is to use Monte Carlo integration.

There are two natural ways to implement the Monte Carlo procedure. In the first, note that since the integrand is a continuous function taking only positive values, the integral represents the area of the region that lies underneath this function, above the x-axis, and between the vertical lines $x=0$ and $x=2$. This region is inscribed in a rectangle having the horizontal line $y=1$ as its upper boundary (see Figure 1). Now imagine throwing a very large number of darts at this rectangle. Assume that each dart is equally likely to land anywhere within this rectangle, independently of every other dart. As the number of darts increases, the proportion that land in the region representing the integral will approach the proportion of the rectangle's area that lies within the region representing the integral. Since the area of the rectangle is two, the integral is estimated by twice the proportion of darts that fall under the curve. Information about the probable error of the estimate can easily be obtained from the fact that the number of darts that fall under the curve is a binomial random variable.

A second way to employ Monte Carlo for the integration problem above is to randomly pick a large number of values of x between 0 and 2 and evaluate the function $\left(x^3+1\right)^{-1/2}$ for each of these. The definite integral above is then esti-

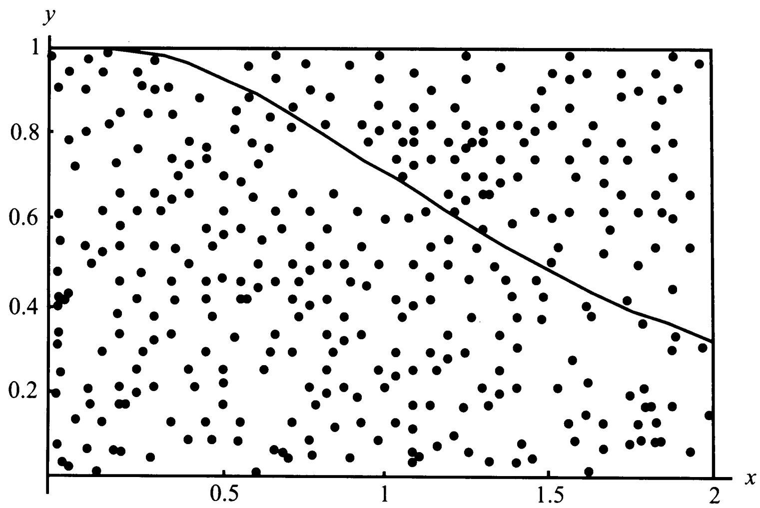

Figure 1. *A Monte Carlo approximation of* $\int_0^2 (x^3+1)^{-1/2}dx$.

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mated by twice the average of the function values obtained. This follows from the calculus result that the average of an integrable function $f(x)$ over the integral $[a,b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Estimating *e* by Monte Carlo

An interesting application of the Monte Carlo method is to obtain an estimate of the transcendental number e. Pirooz Mohazzabi, a physicist, recently described several ways this can be done [1].

One is based on the famous problem of the secretary who places k letters into k addressed envelopes but forgets to check that each goes into the correct envelope. If each letter is placed into a randomly selected envelope, what is the chance that no envelope will contain the correct letter? In the case of three envelopes e_1, e_2, and e_3 and three corresponding letters l_1, l_2, and l_3, the six equally likely possibilities are listed below in columns:

e_1,l_1	e_1,l_1	e_1,l_2	e_1,l_2	e_1,l_3	e_1,l_3
e_2,l_2	e_2,l_3	e_2,l_1	e_2,l_3	e_2,l_1	e_2,l_2
e_3,l_3	e_3,l_2	e_3,l_3	e_3,l_1	e_3,l_2	e_3,l_1

There are two cases with no match (called *derangements*), thus for $k=3$ the chance of no match is $2/6=.33\overline{3}$. For $k=4$ the reader can easily verify that the chance of no match is $9/24=.375$. Surprisingly, as k increases the chance of no match does not grow or shrink but remains nearly constant and in fact converges to $1/e \approx .368$. In fact, the formula for the probability of no match for general k can be shown by an inclusion-exclusion argument to be just the Maclaurin polynomial of order k for $1/e$, $1-1+1/2!-1/3!+\cdots+(-1)^k/k!$. Thus one could estimate e by randomly permuting the integers $1, 2, \ldots, k$ a large number of times for a fairly big value of k, then finding the reciprocal of the proportion of outcomes that yield derangements.

A second method is equivalent to tossing n darts at a board partitioned into n equally likely target regions. The chance that a given region is not hit by any of the darts is $(1-1/n)^n$, which approximates $1/e$ when n is large. So e can be estimated by n divided by the number of regions with no hits.

A third method simulates shaking grains of salt out of a salt shaker. Suppose that on each shake any given grain has probability p of coming out of the shaker. If p is a number of the form $1/n$, then after n shakes each grain has probability $(1-p)^n \approx 1/e$ (for n large) of not having come out. Thus if S is the initial number of grains in the salt shaker and S_n is the number still in it after the nth shake, e can be estimated by S/S_n.

How well does the Monte Carlo method work? Although highly versatile, often it does not achieve the same level of accuracy as other methods. This is because the rate of convergence of Monte Carlo estimates to the exact values generally occurs at the rather slow rate of $1/\sqrt{n}$. A striking example of worst case performance occurs for the famous Buffon's needle problem, perhaps the first Monte Carlo procedure in history.

The problem involves dropping a needle onto a floor with parallel lines spaced l apart. It is not too hard to show that if the length of the needle is l, then the chance that the needle will intersect one of the lines is $2/\pi$. Thus π can be estimated by $2/\hat{p}$, where $\hat{p}$ is the proportion of needles that intersect a line. How accurately does this estimate π? Readers familiar with confidence intervals are invited to derive an approximate 95% confidence interval based on $2/\hat{p}$. Doing so reveals that about 8.65×10^{2d} tosses are required in order to have a 95% chance that the estimate of π is correct to d digits of accuracy. This means that one needs about 8,650 tosses to be reasonably certain of obtaining 3.1, about 865,000 tosses to obtain 3.14, and about 8.65 billion tosses to obtain 3.1416!

Using Probability to Prove Results in Geometry

While the above methods illustrate how probability can be used in the empirical side of other branches of mathematics, probability can contribute to the theoretical aspect as well. Here is one example: Suppose 10% of the surface of a sphere is white, while the rest is black. Prove that there is a cube inscribed in the sphere such that all eight of its vertices are black.

The proof relies on the *Bonferroni Inequality*: Let $A_1,\ldots,A_k$ be a set of events. Then P(at least one of the A_i's occurs) $\le \sum P(A_i)$ where $P(\cdot)$ represents the probability of the enclosed event. (Do you see why Bonferroni's Inequality is true?) Now suppose we choose a random inscribed cube. The probability that any given corner is white is .1, so the probability that *at least* one corner is white is at most .8. Thus the probability that none are white, i.e., all are black, is more than .2. Thus there must be such inscribed cubes.

Here is another good problem: A piece of paper, of any shape (even allowing holes), has an area of ten square centimeters. Show that it can be placed on an integer lattice that has its points spaced one centimeter apart in such a way that at least ten grid points are covered.

To make the proof, suppose that the paper is placed over the grid randomly. Let the random variable X be the number of grid points it covers. It seems logical that the expected value of X is ten. It can be shown (although a rigorous argument is somewhat technical) that this is indeed true. Since it is impossible for a random variable to take only values less than its expectation, there must therefore be at least one case in which $X=10$.

You may wish to try your hand at the following problem: The earth's oceans cover more than one half of the earth's surface (actually about 70%). Use a probability argument to prove that there are two antipodal points (points on opposite ends of a chord through the center of the earth) that are covered by water. You can find the answer to this problem, and see other examples of probabilistic proofs, in the recent paper by Alexander Shen in the *Mathematical Intelligencer* [2]. ■

References

1. Pirooz Mohazzabi, Monte Carlo Estimations of *e*, *American Journal of Physics* 66:2 (February 1998) 138–140.
2. Alexander Shen, Probabilistic proofs, *Mathematical Intelligencer*, Vol. 20, No. 3, 1998.