440A
SCHILLING

SELECTED HW SOLUTIONS, SEC 9.7

9.76 (a) $L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{\theta^2}\right) y_i e^{-y_i/\theta} = \theta^{-2n} \prod_{i=1}^{n} y_i \, e^{-\frac{1}{\theta}\sum y_i}$

$\ln L(\theta) = -2n(\ln\theta) + \ln\left(\prod y_i\right) - \frac{1}{\theta}\sum y_i$

$\frac{\partial \ln L(\theta)}{\partial\theta} = -\frac{2n}{\theta} + 0 + \frac{1}{\theta^2}\sum y_i \overset{S}{=} 0 \Rightarrow \hat{\theta} = \frac{\sum y_i}{2n} = \frac{1}{2}\bar{y} = \frac{1}{2}\,\frac{120+130+128}{3} = 63.$

(b) $E(\hat{\theta}) = \frac{1}{2}E(\bar{Y}) = \frac{1}{2}EY = \frac{1}{2}\alpha\beta = \frac{1}{2}\cdot 2\theta = \theta.$

$V(\hat{\theta}) = V\left(\frac{1}{2}\bar{Y}\right) = \frac{1}{4}V(\bar{Y}) = \frac{1}{4}\frac{\sigma^2}{n} = \frac{1}{4}\frac{\alpha\beta^2}{n} = \frac{2\theta^2}{4n} = \frac{\theta^2}{2n}.$

(d) $\alpha\hat{\theta}^2 = 2(63)^2 = 7938.$

9.78 $L(\sigma^2, \mu_1, \mu_2) = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu_1)^2} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_j-\mu_2)^2}$

$= (2\pi\sigma^2)^{-\frac{m+n}{2}} e^{-\frac{1}{2\sigma^2}\left[\sum(x_i-\mu_1)^2 + \sum(y_j-\mu_2)^2\right]}$

$\ln L = -\frac{m+n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}[\text{" "}]$

$\frac{\partial \ln L}{\partial\sigma^2} = -\frac{m+n}{2}\,\frac{2\pi}{2\pi\sigma^2} + \frac{1}{2(\sigma^2)^2}[\text{" "}] \overset{S}{=} 0 \Rightarrow \ldots \hat{\sigma}^2 = \frac{\sum(x_i-\mu_1)^2 + \sum(y_j-\mu_2)^2}{m+n}$

$\frac{\partial \ln L}{\partial\mu_1} = 0 - \frac{1}{2\sigma^2}\left[-2\sum(x_i-\mu_1) + 0\right] \overset{S}{=} 0 \Rightarrow \sum x_i - n\mu_1 = 0 \Rightarrow \hat{\mu}_1 = \bar{x}$

$\frac{\partial \ln L}{\partial\mu_2} = 0 - \frac{1}{2\sigma^2}\left[0 - 2\sum(y_j-\mu_2)\right] \overset{S}{=} 0 \Rightarrow \sum y_j - n\mu_2 = 0 \Rightarrow \hat{\mu}_2 = \bar{y}$

Solving these three eqn's simultaneously gives $\hat{\sigma}^2 = \frac{\sum(x_i-\bar{x})^2 + \sum(y_j-\bar{y})^2}{m+n}$

9.82 Let $X_1, \ldots, X_n$ be iid Bernoulli r.v.'s representing the men, and let $Y_1, \ldots, Y_n$ be " " " " " women.

Then, similarly to #9.70,

$L(p) = \prod_{i=1}^{n} p_M^{x_i}(1-p_M)^{1-x_i} \prod_{j=1}^{n} p_F^{y_j}(1-p_F)^{1-y_j} = p^{\sum x_i + \sum y_j}(1-p)^{2n - \sum x_i - \sum y_j}$ if $p_M = p_F = p$

(compare to Example 9.14, p.399, for the equivalent problem for one sample)

Thus $L(p) = p^{25+30}(1-p)^{200-25-30} = p^{55}(1-p)^{145}$, so

$\ln L(p) = 55\ln p - 145\ln(1-p)$

$\frac{\partial \ln L(p)}{\partial p} = \frac{55}{p} + \frac{145}{1-p} \overset{S}{=} 0 \Rightarrow \ldots \hat{p} = \frac{55}{200} = .275.$