MATH 440A
SCHILLING

SELECTED HW SOLUTIONS (SEC. 10.10)

10.76 Refer to Exercise 10.2. Since the rejection region is $Y \leq 12$, the power is $1 - \beta = P(Y \leq 12|p)$. To calculate power use Table 1.

(a) power $= P(Y \leq 12|p = .4) = .979$

(b) power $= P(Y \leq 12|p = .5) = .868$

(c) power $= P(Y \leq 12|p = .6) = .584$

(d) power $= P(Y \leq 12|p = .7) = .228$

The graph showing p on the horizontal axis and power = $1 - \beta$ on the vertical axis is omitted here.

10.79 (a) Refer to Example 10.23 in the text. The uniformly most powerful test is found to be the z test of Section 10.3. That is, reject H_0: $\mu = 7$ if

$$Z = \frac{\bar{Y} - 7}{\sqrt{\sigma^2/20}} = \frac{\bar{Y} - 7}{\sqrt{5/20}} \geq 1.645 \qquad \text{or} \qquad \bar{Y} \geq 1.645\sqrt{.25} + 7 = 7.82$$

(b) The power of the test is $1 - \beta = P(\bar{Y} > 7.82|\mu)$.

For $\mu = 7.5$, $\quad 1 - \beta = P(Z > \frac{7.82 - 7.5}{.5}) = P(Z > .64) = .2611$

For $\mu = 8.0$, $\quad 1 - \beta = P(Z > \frac{7.82 - 8}{.5}) = P(Z > -.36) = .6406$

For $\mu = 8.5$, $\quad 1 - \beta = P(Z > \frac{7.82 - 8.5}{.5}) = P(Z > -1.36) = .9131$

For $\mu = 9.0$, $\quad 1 - \beta = P(Z > \frac{7.82 - 9}{.5}) = P(Z > -2.36) = .9909$

(c) The graph is omitted here.

10.80 $\alpha = .05 \Rightarrow$ Reject $H_0 \Leftrightarrow \frac{\bar{Y}-7}{\sigma/\sqrt{n}} > 1.645 = z_{.05}$. Then the power for $\mu = 8$ is $P_{\mu=8}\left(\frac{\bar{Y}-7}{\sigma/\sqrt{n}} > 1.645\right) = P_{\mu=8}\left(\bar{Y} > 7 + 1.645\frac{\sigma}{\sqrt{n}}\right)$

$$= P_{\mu=8}\left(\frac{\bar{Y}-8}{\sigma/\sqrt{n}} > \frac{7+1.645\sigma/\sqrt{n}}{\sigma/\sqrt{n}}\right) = P\left(Z > 1.645 - \frac{1}{\sigma/\sqrt{n}}\right) = P\left(Z > 1.645 - \frac{\sqrt{n}}{\sqrt{5}}\right).$$

(using $\sigma^2 = 5$)

We need $1.645 - \frac{\sqrt{n}}{\sqrt{5}} = -0.84$, since $P(Z > -0.84) \simeq .80$.

This gives $n = 30.88 \Rightarrow$ use $n \geq 31$.

10.83 (a) $L(\theta) = \left(\frac{1}{2\theta^3}\right)^4 \prod_{i=1}^{4} y_i^2 e^{-\frac{1}{\theta}\sum_{i=1}^{4} y_i} \Rightarrow LR = \frac{L(\theta_a)}{L(\theta_0)} = \left(\frac{\theta_0^3}{\theta_a^3}\right)^4 e^{\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum y_i}$.

We reject H_0 when LR is large; since $\theta_a > \theta_0$, LR is an increasing fn. of $\sum_{i=1}^{4} y_i$, hence we reject when $\sum_{i=1}^{4} y_i$ is large. To determine the rejection region, we must find the distribution of $\sum_{i=1}^{n} Y_i$ when H_0 is true. Now the Y_i's are each gamma r.v.'s (see back cover), and can be transformed into chi-square r.v.'s $U_i = 2Y_i/\theta_0$ (see p. 285: $y = h^{-1}(u) = \frac{\theta_0}{2}u \Rightarrow \frac{dy}{du} = \frac{\theta_0}{2} \Rightarrow f_U(u) = \frac{1}{2\theta_0^3}\left(\frac{\theta_0}{2}u\right)^2 e^{-\frac{\theta_0 u}{2\theta_0}} \left|\frac{\theta_0}{2}\right|$ $= \frac{1}{16}u^2 e^{-u/2} = \chi^2_{6\ d.f.}$ (see back cover).) Thus $\sum_{i=1}^{4} U_i \sim \chi^2_{6+6+6+6}$, i.e., we reject $H_0 \Leftrightarrow \sum_{i=1}^{4} \frac{2Y_i}{\theta_0} > \chi^2_{24,\alpha}$.

(b) Since the test depended only on the fact that $\theta_a > \theta_0$ and not on the specific value of θ_0, it is the same $\forall\ \theta > \theta_0 \Rightarrow$ UMP.

10.89 (a) $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-y_i/\theta} = \theta^{-n} e^{-\frac{1}{\theta}\sum y_i} \Rightarrow LR = \left(\frac{\theta_0}{\theta_a}\right)^n e^{\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum y_i}$

For $\theta_a < \theta_0$, LR is a decreasing function of $\sum y_i$ since $\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right) < 0$. So we reject when $\sum y_i$ is small. The distribution of $\sum_{i=1}^{n} Y_i$ is gamma with parameters n and θ (this is easy to show using moment generating functions), so we reject H_0 when $\sum y_i$ falls in the lowest α of the gamma (n, θ_0) distribution.

(b) The test in (a) does not depend on the specific value of θ_a, only on the fact that it is $< \theta_0$. Thus the test is UMP for $H_0: \theta = \theta_0$ vs. $H_a: \theta < \theta_0$.