MATH 440 A
SCHILLING

SOLUTIONS & ADDITIONAL REMARKS FOR LAB #2

1. $L(y_1,\dots,y_n;\theta) = (2T)^{-n} e^{-\frac{1}{T}\sum|y_i-\theta|}$; to maximize L we must minimize $\sum|y_i-\theta|$. We can use calculus, as follows:
$$|y_i-\theta| = \begin{cases} y_i-\theta & \text{if } y_i>\theta \\ \theta-y_i & \text{if } y_i<\theta \end{cases} ; \text{ hence } \frac{d}{d\theta}|y_i-\theta| = \begin{cases} -1 & \text{if } y_i>\theta \\ 1 & \text{if } y_i<\theta \end{cases}.$$
Thus $$\frac{d}{d\theta}\sum|y_i-\theta| = \sum_{i:y_i>\theta} 1 + \sum_{i:y_i<\theta}(-1) = \#(y_i>\theta)-\#(y_i<\theta).$$

This derivative will be 0 precisely when θ is the (a) median, by the definition of the median.

5. $\ln f(y;\theta) = -\ln(2T) - \frac{1}{T}|y-\theta|$; hence $\frac{d\ln f}{d\theta} = \begin{cases} -1/T & \text{if } y>\theta \\ 1/T & \text{if } y<\theta \end{cases}$
Hence $I(\theta) = \dfrac{1}{nE\left(\frac{d\ln f}{d\theta}\right)^2} = \dfrac{1}{n\left(\frac{1}{T^2}\right)} = T^2/n$, which $= Var(\tilde{Y})$ (see #2 below).

(Note: $\frac{d\ln f}{d\theta}$ does not exist at $y=\theta$; however, since $P(Y=\theta)=0$ since Y is a continuous r.v., this has no effect on $E\left(\frac{d\ln f}{d\theta}\right)^2$.)

2. $$Var\,\bar{Y} = \frac{\sigma^2}{n} = \frac{1}{n}E(Y-\mu)^2 = \frac{1}{n}\int_{-\infty}^{\infty}(y-\theta)^2\cdot\frac{1}{2T}e^{-\frac{1}{T}|y-\theta|}dy \text{ (since } \mu=\theta)$$
$= \frac{2}{n}\int_0^\infty \cdots$ since the integrand is a symmetric (even) function;
let $u = \frac{y-\theta}{T}$, so $du = \frac{dy}{T}$, $\Rightarrow Var\,\bar{Y} = \frac{2}{n}\int_0^\infty (Tu)^2\cdot\frac{1}{2T}e^{-|u|}\cdot T\,du$
$= \frac{T^2}{n}\int_0^\infty u^2e^{-u}du = \frac{T^2}{n}\Gamma(3) = \frac{T^2}{n}\cdot 2! = \frac{2T^2}{n}$ (recall the gamma fn., p.165)

$Var\,\tilde{Y} = \dfrac{1}{4nf^2(\theta)}$, where $f(\theta)$ means $f(y;\theta)$ evaluated at $y=\theta$,
$= 1/4n\cdot\left(\frac{1}{2T}e^{-\frac{1}{T}|\theta-\theta|}\right)^2 = 1/4n(1/2T)^2 = \frac{T^2}{n}$.

3. $Eff(\tilde{Y},\bar{Y}) = Var\,\bar{Y}/Var\,\tilde{Y} = \dfrac{2T^2/n}{T^2/n} = 2$. The mean requires twice as much data to do as well as the median here. A consequence is that $\tilde{Y}$ yields narrower CI's (see #4).

4. We use the formula $\hat{\theta} \pm 2\,\hat{\sigma}_{\theta}$ to obtain

$$\bar{Y} \pm 2\sqrt{\frac{2\hat{\tau}^2}{n}} = 110.5 \pm 2\sqrt{\frac{2\,(100)^2}{15}} = 110.5 \pm 73.0$$

and $$\tilde{Y} \pm 2\sqrt{\frac{\hat{\tau}^2}{n}} = 91 \pm 2\sqrt{\frac{100^2}{15}} = 91 \pm 51.6.$$

As expected, the interval based on $\tilde{Y}$ is narrower.

5. $\ln f(y;\theta) = -\ln(2\tau) - \frac{1}{\tau}|y-\theta|$, thus

$$\frac{\partial \ln f(y;\theta)}{\partial \theta} = \begin{cases} 1/\tau & \text{if } \theta < y, \\ -1/\tau & \text{if } \theta > y. \end{cases}$$

Thus $$I(\theta) = \frac{1}{nE\left[\left(\frac{\partial \ln f}{\partial \theta}\right)^2\right]} = \frac{1}{nE\left(\left(\pm\frac{1}{\tau}\right)^2\right)} = \frac{1}{n\left(\frac{1}{\tau}\right)^2} = \frac{\tau^2}{n}.$$

Thus $\tilde{Y}$ does achieve the Cramer-Rao lower bound. Hence it is the MVUE of θ.

6. $$L(\tau) = \prod_{i=1}^{n} (2\tau)^{-1} e^{-\frac{1}{\tau}\Sigma|y_i-\theta|} = (2\tau)^{-n} e^{-\frac{1}{\tau}\Sigma|y_i-\theta|}$$

$$\Rightarrow \ln L(\tau) = -n\ln(2\tau) - \frac{1}{\tau}\Sigma|y_i-\theta|$$

$$\Rightarrow \frac{\partial \ln L(\tau)}{\partial \tau} = -\frac{n}{\tau} + \frac{1}{\tau^2}\Sigma|y_i-\theta| \stackrel{s}{=} 0 \Rightarrow \hat{\tau} = \frac{\Sigma|y_i-\theta|}{n}.$$

Since θ is unknown we must use its m.l.e. in place of θ to obtain

$$\hat{\tau} = \frac{\Sigma|y_i-\tilde{y}|}{n} = \frac{1}{15}\left[(209-91)+(341-91)+\ldots+(-13-91)\right]$$

$$= \frac{1}{15}(1500) = 100.$$