MATH 440B
SCHILLING

HW SOLUTIONS (BOOTSTRAP)

8.34 (a) Write μ for μ_0, σ^2 for σ_0^2. Recall from Math 440A that $\hat{\mu} = \bar{x}$ and that the sampling distribution of $\bar{X}$ is $N(\mu, \sigma^2/n)$. Since bootstrapping uses sample statistics in place of parameters ($\hat{\theta}$ for θ), the bootstrap estimate of this distribution is $N(\hat{\mu}, \hat{\sigma}^2/n)$. An alternative justification is that the bootstrap sample means $\bar{x}_j^*$ are based on samples from the discrete distribution that puts probability $1/n$ on each of $x_1, \ldots, x_n$; this distribution has mean $\sum_{i=1}^{n} x_i \cdot \frac{1}{n} = \bar{x}$ and variance $\sum_{i=1}^{n} (x_i - \bar{x})^2 \cdot \frac{1}{n} = \hat{\sigma}^2$. Furthermore, the distribution will be $\approx$ normal by the CLT if n is large.

(b) The distribution of $\hat{\mu} - \mu = \bar{X} - \mu$ is $N(0, \sigma^2/n)$. Replacing σ^2 by its estimate $\hat{\sigma}^2$ gives the desired result, $N(0, \hat{\sigma}^2/n)$.

(c) Pivoting $P(a \le \hat{\mu} - \mu \le b) = 1 - \alpha$ yields $P(\hat{\mu} - b \le \mu \le \hat{\mu} - a) = 1 - \alpha$, where a & b are obtained from the distribution given in part (b) above. Specifically, a & b are the $\frac{\alpha}{2}$-quantile and the $(1 - \frac{\alpha}{2})$-quantile of that distribution, $a = -z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}$, $b = z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}$. Thus the bootstrap CI for μ is $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}}$, as compared to the classical CI $\bar{x} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$, where $s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$.

8.37 The CI $(\hat{\theta} - b, \hat{\theta} - a)$ is estimated with bootstrapping by choosing a and b to be the lower and upper quantiles of the bootstrap distribution of $\theta^* - \hat{\theta}$. The quantiles of θ^* are just shifted by $\hat{\theta}$: $\underline{\theta} = a + \hat{\theta}$, $\bar{\theta} = b + \hat{\theta}$. Thus $a = \underline{\theta} - \hat{\theta}$, $b = \bar{\theta} - \hat{\theta}$. Substituting these values into the CI above gives the result.

8.38 Symmetry $\Rightarrow \hat{\theta} = \frac{1}{2}(\underline{\theta} + \overline{\theta})$. Substituting this $\hat{\theta}$ into the CI given in #33 gives the desired result.

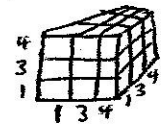
8.39 Using $P(\underline{\delta} \le \hat{\theta}/\theta \le \overline{\delta}) = 1-\alpha$ gives $P(\hat{\theta}/\overline{\delta} \le \theta \le \hat{\theta}/\underline{\delta}) = 1-\alpha$ $\Rightarrow (\hat{\theta}/\overline{\delta}, \hat{\theta}/\underline{\delta})$ is a level $1-\alpha$ CI for θ. Then $\overline{\delta}$ and $\underline{\delta}$ are estimated by the quantiles of the bootstrap distn of $\theta^*/\hat{\theta}$.

8.40 (a) The estimate is the proportion of bootstrap samples in which $|\theta^* - \hat{\theta}| > .01$.

(b) The estimate is $\frac{1}{B}\sum_{j=1}^{B} |\theta^* - \hat{\theta}|$.

(c) Find $\hat{\Delta}$ such that the proportion of bootstrap samples in which $|\theta^* - \hat{\theta}| > \Delta$ is .5.

10.31 (b) Each of the 27 possible bootstrap samples is represented by a cell in the cube shown:



(c) See book answer to (b), correct $p(8/3) = 6/27$, $p(3) = 4/27$, and add $x = 4$, $p(4) = 1/27$.

(d) Using the distribution of $\bar{X}$ indicated in (c), we obtain $\mu_{\bar{X}} = \sum x p(x) = 2.67 \Rightarrow \sigma_{\bar{X}} = \sqrt{\sum (x - \mu_{\bar{X}})^2 p(x)} = 0.72$. The usual estimated standard error is $s_{\bar{X}} = s/\sqrt{n}$, where s is the sample s.d. of the x_i's 1, 3 & 4. We obtain $s = 1.53 \Rightarrow s_{\bar{X}} = \frac{1.53}{\sqrt{3}} = 0.88$, similar to $\sigma_{\bar{X}}$.

10.32 Draw a large number B of bootstrap samples, and compute the MAD for each one. The distribution of the B MAD's will approximate the sampling distribution of the MAD.

Computer Homework

#6 CI for $\mu = \bar{X} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} = (161.3, 174.1)$.

CI for $\sigma^2 = \left(\frac{(n-1)s^2}{\chi^2_U}, \frac{(n-1)s^2}{\chi^2_L}\right) = (37.8, 266.4)$.

These values should be at least somewhat close to those obtained by bootstrapping