Since the calculation above works for any value of $M > 0$, it is clear that you should exchange Envelope 1 for Envelope 2. Then the same argument implies that you should now swap again—back to Envelope 1!

A Switch in Time Pays Fine?

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One of the most well known probability brainteasers of recent years is the Three Doors problem, modeled after the old television game show *Let's Make a Deal*. The contestant is to choose one of the three doors, and wins whatever is behind it. One of the doors conceals a valuable prize, for instance a new sports car, while behind each of the other two doors is a booby prize. Once the contestant selects a door, the host of the show opens not that door but a different door, revealing (say) a goat. The contestant is then offered the opportunity to switch from the door she selected to the remaining unopened door. Usually the contestant does not change doors, and in fact most people are convinced that each of the two doors has an equal chance of hiding the sports car. Surprisingly, however, the best strategy is to switch doors.

This problem has received so much attention that most people in the mathematics community are familiar with it, and I will not discuss it further here. (If you have not heard it before, see if you can reason out why switching is the best strategy. You may also wish to set up a simulation of the game and try it a large number of times with a friend. Or see Ed Barbeau's 1993 article "The Problem of the Car and Goats" in the *College Mathematics Journal*.) Instead I want to discuss another strange brainteaser that has become popular in the last few years and also involves the question of switching choices.

The problem appeared in print in a 1995 article in *Mathematics Magazine* by Steven J. Brams and D. Marc Kilgour called "The box problem: to switch or not to switch," and goes as follows: Suppose that there are two envelopes, one with $X > 0$ dollars inside and the other containing $2X$ dollars. You select one of the envelopes—call this Envelope 1—and examine its contents. Then you are given the opportunity to switch to the other envelope, Envelope 2. Should you switch?

You may think that the offer to change envelopes is ridiculous—without any way of knowing the value of X or which envelope is which, one choice seems as good as the other, so switching gives no better or worse prospects than not switching. But consider the following argument: Suppose that the amount of money in Envelope 1 turns out to be M dollars. Then with probability 0.5 Envelope 2 contains $M/2$ dollars, while with probability 0.5 Envelope 2 contains $2M$ dollars. If you change to Envelope 2, then, your expected winnings are $0.5(M/2) + 0.5(2M) = 1.25M$ dollars. This calculation indicates that it is always best to switch, as you would expect to receive 25% more money in the long run with this strategy.

But how can this be? After all, had you been presented with Envelope 2 to begin with, the same logic would tell you to exchange it for Envelope 1. In fact, consider a variation on the above game in which you are not even allowed to open an envelope before deciding whether to keep it. Since the calculation above works for any value of $M > 0$, it is clear that you should exchange Envelope 1 for Envelope 2. Then the same argument implies that you should now swap *again*—back to Envelope 1!

Attempts to resolve this paradox involve modeling the contents of the two envelopes X and $2X$ as random variables and enumerating the possible values of X and their corresponding probabilities. This probability distribution may be specified as part of the game, or it may represent an expression of your subjective belief about the likely values in the envelopes.

For example, suppose you know that the benefactor who loaded the envelopes chose $(X, 2X)$ to be one of the setups ($2, $4), ($10, $20) and ($100, $200), with probability 1/3 for each. If you inspect the contents of Envelope 1 then you will know with certainty whether Envelope 2 contains more or less money, so your strategy is trivial. In the case when you are not allowed to open Envelope 1, the expected value computation above no longer applies, and the correct calculation shows that the paradox vanishes—both switching and staying now yield the same expected payoff. In fact, with any finite set of values with which to load the envelopes and any set of associated

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probabilities, the paradox disappears, as the expected payoff is the same with both envelopes.

Yet the envelope problem cannot be dispensed with quite so easily, as the following example from Brams and Kilgour illustrates: Suppose that there are an infinite number of possible values of X, specifically \$1, \$2, \$4, \$8, …, with probability $(1/3)\cdot(2/3)^k$ that X equals $2k$ dollars, for $k = 0, 1, 2, \ldots$. You can check that these probabilities add to one, so this is a legitimate probability distribution.

The table below shows the possible envelope contents and their associated probabilities:

Contents of Envelope 1	Contents of Envelope 2	Probability
\$1	\$2	1/6
\$2	\$1	1/6
\$2	\$4	1/9
\$4	\$2	1/9
\$4	\$8	2/27
\$8	\$4	2/27
\$8	\$16	4/81
⋮	⋮	⋮

If Envelope 1 contains \$1, swapping envelopes would improve your payoff to \$2. Now suppose Envelope 1 contains \$2. To determine whether switching is a good idea, we must determine your conditional expected payoff. This is done by renormalizing the probabilities in the second and third rows of the above table to sum to one: $(1/6) \to (1/6) \div (1/6 + 1/9) = 3/5$, $(1/6) \to (1/9) \div (1/6 + 1/9) = 2/5$. That is, there is a 60% chance that you will end up with a lower amount (\$1) if you switch, and a 40% chance of receiving a higher amount (\$4). Although changing envelopes is more likely to reduce your payoff than to increase it, your expected payoff in this case is \$1(3/5) + \$4(2/5) = \$11/5 = \$2.20, which is 10% higher than the \$2 you currently have.

It is easy to check that if Envelope 1 contains any amount greater than \$1, swapping envelopes yields a conditional expected payoff that is 10% higher than the amount you'll receive if you don't switch. Amazingly, then, it seems that switching is always the right strategy! We needn't even open Envelope 1.

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had calculated (October 22, 4004 BC), among other errors. Arbuthnott concluded that there was less variation in the sex ratio than would occur by chance when indeed there was too much and asserted falsely that constancy of the sex ratio would hold not just for the 82 years in London which produced the data, but all over the world and for all time when indeed we now known that there is spatial and temporal variation in the sex ratio. Yet the value of their insights overwhelmed their errors. Progress is not made as perfection, but as small truths in a sea of errors. ■

For Further Reading

Graunt's book and Arbuthnott's paper are available on the internet at `www.ac.wwu.edu/~stephan/Graunt/graunt.html` and `www.taieb.net/auteurs/Arbuthnot/arbuth.html`, respectively. Anders Hald's *A History of Probability and Statistics and Their Applications before 1750* has an extensive discussion of these two manuscripts. Charles Creighton's *A History of Epidemics in Britain* has extensive data on mortality for this period if one wishes to analyze such data. Vanessa Harding's paper "The population of London 1550-1700: a review of the published evidence" which appeared in *London Journal* in 1990 discusses how little we know about how many people lived in London.

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Obviously something is haywire here. If switching always increases your conditional expected payoff, then your overall expected payoff must also be increased by switching. And you could make it even greater by switching back again, and again, and again...! We seem to have a situation where A > B > A .... If you would like to find the bug in this argument yourself, then don't read on just yet....

OK, ready to dispense with the Emperor's clothes? Then let's look more closely at the calculation of the expected payoffs under the stay and switch strategies.

For the stay strategy, your expected payoff with Envelope 1 is:

$$(\$1)(1/6)+(\$2)(1/6+1/9)+(\$4)(1/9+2/27)+\ldots$$

$$=(\$1)\frac{1}{6}+\sum_{k=1}^{\infty}\left(\$2^k\right)\left[\frac{1}{6}\left(\frac{2}{3}\right)^{k-1}+\frac{1}{6}\left(\frac{2}{3}\right)^{k}\right]$$

$$=\$\frac{1}{6}\left[1+\frac{5}{2}\sum_{k=1}^{\infty}\left(\frac{4}{3}\right)^{k}\right].$$

Since this expression contains a divergent geometric series, your expected payoff with the original envelope is infinite! Thus, saying that changing envelopes gives a higher expected payoff makes no sense, as both expected payoffs are infinite. ■