9.2 a. Since $E(Y_i) = \mu$, $i = 1, 2, \ldots, n$, then

$$E(\hat{\mu}_1) = \tfrac{1}{2}(2\mu) = \mu \qquad E(\hat{\mu}_2) = \tfrac{1}{4}\mu + \frac{(n-2)\mu}{2(n-2)} + \tfrac{1}{4}\mu = \mu \qquad E(\hat{\mu}_3) = \frac{n\mu}{n} = \mu$$

b. Further,

$$V(\hat{\mu}_1) = \tfrac{1}{4}(2\sigma^2) = \frac{\sigma^2}{2} \quad V(\hat{\mu}_2) = \frac{2}{16}\sigma^2 + \frac{(n-2)\sigma^2}{4(n-2)^2} = \frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)} \quad V(\hat{\mu}_3) = \frac{\sigma^2}{n}$$

Hence the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_1$ is

$$\frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = \frac{\left(\frac{\sigma^2}{2}\right)}{\left(\frac{\sigma^2}{n}\right)} = \frac{n}{2}$$

and the efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_2$ is

$$\frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = \frac{\frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)}}{\frac{\sigma^2}{n}} = \frac{n}{8} + \frac{n}{4(n-2)} = \frac{n^2}{8(n-2)}$$

9.6 $V(\hat{\lambda}_1) = V\left[\frac{1}{2}(Y_1 + Y_2)\right] = \left(\frac{2}{4}\right)\lambda = \frac{\lambda}{2}$

$V(\hat{\lambda}_2) = V(\overline{Y}) = \frac{\lambda}{n}$

$\text{eff}(\hat{\lambda}_1, \hat{\lambda}_2) = \frac{2}{n}$

9.7 Since Y is exponential, $E(Y) = \theta$ and $V(Y) = \theta^2$.

$V(\hat{\theta}_1) = \theta^2$, since it was given that $\hat{\theta}_1$ is an unbiased estimator and $\text{MSE}(\hat{\theta}_1) = \theta^2$.

$V(\hat{\theta}_2) = V(\overline{Y}) = \frac{\theta^2}{n}$

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\left(\frac{\theta^2}{n}\right)}{\theta^2} = \frac{1}{n}$$

9.8 a. For $\hat{\theta} = \overline{Y}$, $V(\hat{\theta}) = V(\overline{Y}) = \frac{\sigma^2}{n}$. The Cramer-Rao lower bound is calculated by using

$$\ln f(y) = c - \frac{(y-\mu)^2}{2\sigma^2}$$

where c is a constant not involving μ. Then we have

$$\frac{\partial \ln f(y)}{\partial \mu} = \frac{y-\mu}{\sigma^2} \qquad \text{and} \qquad \frac{\partial^2 \ln f(y)}{\partial^2 \mu} = -\frac{1}{\sigma^2}$$

Finally,

$$I(\mu) = \frac{1}{\left[nE\left(\frac{1}{\sigma^2}\right)\right]} = \frac{\sigma^2}{n}, \text{ which is equal to } V(\overline{Y}), \text{ so that } \overline{Y} \text{ is efficient.}$$

b. Similar to part (a) with $V(\overline{Y}) = \frac{\sigma^2}{n} = \frac{\lambda}{n}$ and $p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$. Calculate

$$\ln p(y) = y \ln \lambda - \lambda - \ln y! \quad \text{and} \quad \frac{\partial \ln p(y)}{\partial \lambda} = \frac{y}{\lambda} - 1 \quad \text{and} \quad -\frac{\partial^2 \ln p(y)}{\partial^2 \lambda} = \frac{y}{\lambda^2}$$

Finally,

$$I(\mu) = \frac{1}{\left[nE\left(\frac{y}{\lambda^2}\right)\right]} = \frac{\lambda}{n}, \text{ which equals } V(\overline{Y}), \text{ so that } \overline{Y} \text{ is efficient for } \lambda.$$