Yet another failed attempt to prove the Irrationality of $\zeta(5)$

Abstract

1 Introduction

The irrationality of the values of the Riemann zeta function at even integers has been known since Euler. However, for odd integers very little is known. A breakthrough came in 1978 when R. Apery presented his proof that $\zeta(3)$ is irrational [2]. Shortly thereafter F. Beukers presented a much more accessible proof of this result [4]. In the past three decades many papers have appeared studying the irrationality of $\zeta(2n+1)$ using Beukers type integrals and the closely related Sorokin type integrals. In many of them linear combinations of the values $\zeta(n)$ for $n > 2$ over $\mathbb{Q}$ are examined. These yield to lower bounds of the dimensions of the $\mathbb{Q}$ vector spaces spanned by values of the zeta function. T. Rivoal and K. Ball showed that infinitely many of the values $\zeta(2n+1)$ are irrational [10, 3], and W. Zudilin later showed that at least one of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational [14]. Works by T. Rivoal, S. Fischler, Yu. Nesterenko, C. Viola, and W. Zudilin improved these techniques and results (see, for example, [5, 7, 11, 9, 13]). But to date, the irrationality for a specific value of $\zeta(2n+1)$ other than $\zeta(3)$ has not been shown. In this note we explore $\zeta(5)$ via two different approaches using Beukers type integrals. While both approaches ultimately fail to produce the desired result, the nature of this failure comes at the opposite end of the arguments. This may give some information about how to approach this problem. Our approaches are based on the following easily proved identity:

$$\zeta(5) = \int_{(0,1)^4} \frac{-\log x_1x_2x_3x_4}{1 - x_1x_2x_3x_4} dx_1dx_2dx_3dx_4 = \frac{1}{6}\int_{(0,1)^2} \frac{-(\log x_1x_2)^3}{1 - x_1x_2} dx_1dx_2 \tag{1}$$

In section 2 of this note we will use the double integral to show that a positive linear expression involving only $\zeta(5)$ and integers can be expressed by a Beukers type integral. We believe that this approach is new and can offer some valuable insights. Unfortunately, this integral does not give the desired upper bound which allows one to show that the expression will go to zero as N increases. This

approach was inspired by J. Sondow's work on irrationality conditions for Euler's constant [12], and D. Huylebrouck's paper [8].
In section 3 we will use an approach based on the quadruple integral. While this approach yields excellent upper bounds, one only gets them for a positive linear combination of $\zeta(3)$, $\zeta(4)$, and $\zeta(5)$ over $\mathbb{Z}$.

2 Take 1, the double integral approach

The main result here is

Proposition 1 *For every $N \in \mathbb{N}$ there exist integers A_N and B_N such that*

$$0 < A_N\zeta(5) + B_N = \int_{(0,1)^2} \frac{(\log x_1x_2)^3 p_N(x_1)p_N(x_2)}{1 - x_1x_2} dx_1dx_2 \; d_N^5, \tag{2}$$

where $d_N = \mathrm{lcm}(1, 2, \ldots, N$, and $p_N(x)$ denotes the Legendre polynomial of degree N over $[0, 1]$.

We split the proof of this result into two Lemmas:

Lemma 1 *Let $r, s \in \mathbb{N}$. Let $d_r = \mathrm{lcm}(1, 2, \ldots r)$. We have:*

$$\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 x_1^r x_2^s}{1 - x_1x_2} dx_1dx_2 = \begin{cases} 24\left(\zeta(5) - \sum_{k=1}^{r} \frac{1}{k^5}\right) & \text{if} \quad r = s \\ q_r \in \mathbb{Q} & \text{if} \quad r > s \end{cases} \tag{3}$$

where $d_r q_r \in \mathbb{Z}$.

Proof: The proof follows the proof of Lemma 7.7.3 in [1] We start by noting that

$$(\log x_1x_2)^3 = \frac{d^3}{d\epsilon^3}(x_1x_2)^\epsilon\bigg|_{\epsilon=0}.$$

Replacing the logarithm in the integral by the expression on the right hand side of this last equation and expanding the geometric series one has

$$\begin{aligned} \int_{(0,1)^2} \frac{-(\log x_1x_2)^3 x_1^r x_2^s}{1 - x_1x_2} dx_1dx_2 &= -\frac{d^3}{d\epsilon^3} \sum_{k=0}^{\infty} \int_{(0,1)^2} x_1^{r+\epsilon} x_2^{s+\epsilon} dx_1dx_2 \bigg|_{\epsilon=0} \\ &= -\frac{d^3}{d\epsilon^3} \sum_{k=0}^{\infty} \frac{1}{(k+1+r+\epsilon)(k+1+s+\epsilon)}\bigg|_{\epsilon=0} \end{aligned}$$

If $r > s$, the last sum will collapse and yield

$$-\frac{d^3}{d\epsilon^3}\frac{1}{r-s}\left\{\frac{1}{s+1+\epsilon} + \cdots + \frac{1}{r+\epsilon}\right\}\bigg|_{\epsilon=0}$$

Evaluating the third derivative will give

$$\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 x_1^r x_2^s}{1-x_1x_2} dx_1dx_2 = \frac{6}{r-s}\left\{\frac{1}{(s+1)^4}+\cdots+\frac{1}{r^4}\right\}$$

When $r = s$, we take the third derivative and evaluate at $\epsilon = 0$ to get

$$\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 x_1^r x_2^r}{1-x_1x_2} dx_1dx_2 = 24\sum_{k=0}^{\infty}\frac{1}{(k+1+r)^5} = 24\sum_{k=r+1}^{\infty}\frac{1}{k^5}$$

and the proof of the Lemma is complete.

To continue we show

Lemma 2 *There exist integers A_n and B_N such that*

$$0 < \int_{(0,1)^2} \frac{-(\log x_1x_2)^3 p_N(x_1)p_N(x_2)}{1-x_1x_2} dx_1dx_2 = \left(A_N\zeta(5)+B_N\right) d_N^{-5}$$

Proof: Since the Legendre polynomials $p_N(x_1), p_N(x_2)$ have integral coefficients, the equality follows from the previous lemma. To show the positivity one first eliminates the logarithms by writing

$$\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 p_N(x_1)p_N(x_2)}{1-x_1x_2} dx_1dx_2 = \int_{(0,1)^2} \frac{-(\log x_1x_2)^3(-x_1x_2)^2 p_N(x_1)p_N(x_2)}{(1-x_1x_2)^3} dx_1dx_2$$

and using the identity

$$-\frac{\log x_1x_2}{1-x_1x_2} = \int_0^1 \frac{dz_1}{1-(1-x_1x_2)z_1}$$

three times to get

$$\begin{aligned}
&\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 p_N(x_1)p_N(x_2)}{1-x_1x_2} dx_1dx_2 \\
&\quad = \int_{(0,1)^5} \frac{(1-x_1x_2)^2 p_N(x_1)p_N(x_2)}{(1-(1-x_1x_2)z_1)(1-(1-x_1x_2)z_2)(1-(1-x_1x_2)z_3)} dx_1dx_2dz_1dz_2dz_3.
\end{aligned}$$

The critical part of Beukers' proof is the next sequence of integration by parts, followed by substitution followed by a second integration by parts. The more complicated denominator and extra factor in the numerator complicate this process. An easy induction argument shows that

$$\begin{aligned}
&\frac{\partial^n}{\partial x_1^n}\frac{1}{(1-(1-x_1x_2)z_1)(1-(1-x_1x_2)z_2)(1-(1-x_1x_2)z_3)} \\
&\quad = (-1)^n n! x_2^n \sum_{i+j+k=n} \frac{z_1^i z_2^j z_3^k}{(1-(1-x_1x_2)z_1)^{i+1}(1-(1-x_1x_2)z_2)^{j+1}(1-(1-x_1x_2)z_3)^{k+1}} \\
&\quad = (-1)^n n! x_2^n Q_n.
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\frac{\partial^n}{\partial x_1^n}\frac{(1-x_1x_2)^2}{(1-(1-x_1x_2)z_1)(1-(1-x_1x_2)z_2)(1-(1-x_1x_2)z_3)}\\
&\quad=(1-x_1x_2)^2(-1)^n n! x_2^n Q_n-4n(1-x_1x_2)x_2(-1)^{n-1}(n-1)!x_2^{n-1}Q_{n-1}\\
&\qquad+4\frac{n(n-1)}{2}x_2^2(-1)^{n-2}(n-2)!x_2^{n-2}Q_{n-2}\\
&\quad=(-1)^n n! x_2^n\left((1-x_1x_2)^2Q_n+4(1-x_1x_2)Q_{n-1}+2Q_{n-2}\right).
\end{aligned}
$$

Observe that all terms inside the prentices are positive and it is easily seen that the sums defining Q_n have $(n+1)(n+2)/2$ terms and so the number of terms grows like a second degree polynomial in n. Integrating by parts in our original integral N times will produce

$$
\int_{(0,1)^5}\left((1-x_1x_2)^2Q_N+4(1-x_1x_2)Q_{N-1}+2Q_{N-2}\right)x_2^N x_1^N(1-x_!)^N P_N(x_2)dx_1dx_2dz_1dz_2dz_3
$$

We continue by making the substitutions

$$
w_i=\frac{1-z_i}{1-(x_1x_2)z_i}
$$

for $i=1,2,3$. This has the effect that terms of the form

$$
\begin{aligned}
&\frac{z_1^i z_2^j z_3^k}{(1-(1-x_1x_2)z_1)^{i+1}(1-(1-x_1x_2)z_2)^{j+1}(1-(1-x_1x_2)z_3)^{k+1}}\\
&\quad=\frac{(1-w_1)^i(1-w_2)^j(1-w_3)^k}{(1-(1-x_1x_2)w_1)(1-(1-x_1x_2)w_2)(1-(1-x_1x_2)w_3)}
\end{aligned}
$$

So the new integrand can be written as

$$
\frac{\left((1-x_1x_2)^2R_N+4(1-x_1x_2)R_{N-1}+2R_{N-2}\right)(1-x_2)^N P_N(x_2)}{(1-(1-x_1x_2)w_1)(1-(1-x_1x_2)w_2)(1-(1-x_1x_2)w_3)},
$$

where

$$
R_N=\sum_{i+j+k=N}(1-w_1)^i(1-w_2)^j(1-w_3)^k
$$

Except for the R_N part of this the three terms are similar, and in one case equal to the original integrand. The R_N are unaffected by the integration by parts in x_2. Integration by parts N times in x_2 will yield the following integrand

$$
\begin{aligned}
I_N=\Big(&R_N\left((1-x_1x_2)^2Q_N+4(1-x_1x_2)Q_{N-1}+2Q_{N-2}\right)\\
&+R_{N-1}\left((1-x_1x_2)Q_N+2Q_{N-1}\right)+2R_{N-2}Q_N\Big)\\
&\times\left(x_1(1-x_1)x_2(1-x_2)\right)^N.
\end{aligned}
$$

This integrant is clearly positive, which conludes the proof of the lemma and the proposition.
To prove the irrationality of $\zeta(5)$, one must now show that this integral converges to zero at a rate faster than the exponential growth of d_N^5. From the Prime Number Theorem we know that

$$d_N < e^{(1+\epsilon)N},$$

which yields a growth for d_N^5 which is faster than e^{5N}. To look at the growth of our integrand we write

$$I_N = \frac{J_N}{(1-(1-x_1x_2)w_1)(1-(1-x_1x_2)w_2)(1-(1-x_1x_2)w_3)} (x_1(1-x_1)x_2(1-x_2))^N$$

J_N is sum of q_N terms, where q_N is a polynomial of degree four. The terms of J_N are of the form

$$\frac{(1-w_1)^i(1-w_2)^j(1-w_3)^k w_1^s w_2^t, w_3^u}{(1-(1-x_1x_2)w_1)^s(1-(1-x_1x_2)w_2)^t(1-(1-x_1x_2)w_3)^u},$$

where $i+j+k \in \{N-2, N-1, N\}$ and $s+t+u \in \{N-2, N-1, N\}$. Since it adds only a fixed factor we may assume that $i+j+k = N$ and $s+t+u = N$. Next, observe that all combinations of the sums $i+j+k$ and $s+t+u$ actually appear in J_N so that J_N is invariant under any group action of S_3 which permutes the values of w_1, w_2, and w_3. So if I_N attaines a maximal value at some particular point $(x_1^0, x_2^0, w_1^0, w_2^0, w_3^0)$ it will attain that same value at $(x_1^0, x_2^0, w_{\sigma(1)}^0, w_{\sigma(2)}^0, w_{\sigma(3)}^0)$ for any permutation $\sigma \in S_3$. To continue, let us suppose that the maximal value is attained along the diagonal $w_1 = w_2 = w_3$, for if it is not the actual maximal value is certainly at least as large as the maximum along the diagonal. Replacing the individual values of w_i in J_N by a common w, the terms of J_N all become

$$\frac{(1-w)^N w^N}{(1-(1-x_1x_2)w)^N}$$

which together with the factors $x_1^N(1-x_2)^N x_2^N(1-x_2)^N$ combines to the original integrand in Beukers' proof of the irrationality of $\zeta(3)$. Beukers provided an upper bound for this integrand in the form of

$$C(1+\sqrt{2})^{-4N}.$$

Combining this with the fact that the actual maximum may excede the value on the diagonal we get

Lemma 3 *If there is a real number ρ such that*

$$\int_{(0,1)^2} \frac{-(\log x_1x_2)^3 p_N(x_1)p_N(x_2)}{1-x_1x_2} dx_1 dx_2 \leq K q_N \rho^N,$$

for some constant K and the polynomial q_N obtained in the derivation above, then

$$\rho \geq \frac{1}{(1+\sqrt{2})^4}$$

Since $(1+\sqrt{2})^4 < e^5$, this approach cannot possibly yield a proof for the irrationality of $\zeta(5)$. However, it came as a real surprise to the author that the critical steps of the proof of Lemma 2, namely the sequence of integration by parts, followed by substitution, followed by another integration by parts, still work without returning expressions which are not managable.

3 Take 2, the quadruple integral approach

Another way of trying is to use the quadruple integral representation mentioned in the introduction. This attempt will fail as well, however, the failure will be at a different level of the proof. It will reproduce the known result:

Proposition 2 *There are integer sequences A_N, B_N, C_N, D_N such that*

$$0 < A_N\zeta(5) + B_N\zeta(4) + C_N\zeta(3) + D_N < K3^{-N},$$

for some constant K. Moreover, if $d_N = \operatorname{lcm}(1,\ldots,N)$ then

$$d_N^5|A_N, \qquad d_N^4|B_N, \qquad D_N^3|C_N.$$

As in the previous section we divide the proof of this proposition into several lemmas. Moreover, as we have a quadruple integral to play with we will not use the notations $x_i, i = 1,\ldots,4$, but rather x, y, u, v, as the (x,y) variables are handled very differently than the (u,v) ones.

Lemma 4 *Let s_1, s_2, s_3, and s_4 be nonnegative integers and $r = \max\{s_1,s_2,s_3,s_4\}$. Moreover, let $d_r = \operatorname{lcm}(1,\ldots,r)$. Then*

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{-\log xyuv}{1-xyuv}x^{s_1}y^{s_2}u^{s_3}v^{s_4}\,dxdydudv \tag{4}$$

$$= \begin{cases} 4\left(\zeta(5) - \sum_{k=1}^r \frac{1}{k^5}\right) & \text{if} \quad s_1 = s_2 = s_3 = s_4 = r \\ q_1 \in \mathbb{Q} & \text{if the four exponents are pairwise distinct} \\ p_2\zeta(3) + q_2 & \text{if there are three distinct exponents} \\ p_3\zeta(3) + q_3 & \text{if there are two distinct exponents} \\ & \text{appearing twice each} \\ p_4\zeta(4) + p_4'\zeta(3) + q_4 & \text{if there are two distinct exponents,} \\ & \text{with one appearing three times} \end{cases}, \tag{5}$$

where

$$d_r^5 q_i \in \mathbb{Z} \tag{6}$$

$$d_r^3 p_2, d_r^2 p_3, d_r^2 p_4' \in \mathbb{Z} \tag{7}$$

$$d_r p_4 \in \mathbb{Z} \tag{8}$$

Proof: Following the lines of the previous section we write

$$\log xyuv = \frac{d}{d\epsilon}(xyuv)^{\epsilon}\Big|_{\epsilon=0}$$

and expand the integrand into a power series. After integration we can look at the different cases.
Case 1: $s_1 = s_2 = s_3 = s_4$. In this case the terms of the series are given by

$$\frac{1}{(k+1+s_1+\epsilon)^4}$$

and differentiation and evaluating at $\epsilon = 0$ will yield the expression involing $\zeta(5)$.
Case 2: Four pairwise distinct exponents. In this case we have the typical term

$$\begin{aligned}
&\frac{1}{(k+1+s_1+\epsilon)(k+1+s_2+\epsilon)(k+1+s_3+\epsilon)(k+1+s_4+\epsilon)} \\
&\quad = \frac{1}{(s_2-s_1)(s_3-s_4)}\left(\frac{1}{k+1+s_1+\epsilon} - \frac{1}{k+1+s_2+\epsilon}\right)\left(\frac{1}{k+1+s_3+\epsilon} - \frac{1}{k+1+s_4+\epsilon}\right) \\
&\quad = \frac{1}{(s_2-s_1)(s_3-s_4)}\left(\frac{1}{(k+1+s_1+\epsilon)(k+1+s_3+\epsilon)} - \frac{1}{(k+1+s_2+\epsilon)(k+1+s_3+\epsilon)}\right. \\
&\qquad \left. + \frac{1}{(k+1+s_1+\epsilon)(k+1+s_4+\epsilon)} - \frac{1}{(k+1+s_2+\epsilon)(k+1+s_4+\epsilon)}\right)
\end{aligned}$$

Partial fraction decomposition can be done again resulting in four collapsing sums of the form

$$\frac{1}{(s_2-s_1)(s_4-s_3)(s_i-s_j)}\sum_{k=0}^{\infty}\left(\frac{1}{(k+1+s_j+\epsilon)} - \frac{1}{(k+1+s_i+\epsilon)}\right).$$

Since these sums are finite, differentiation and evaluating the result at $\epsilon = 0$ will result in a rational number with a denominator with the stated properties.
Case 3: If there are three distinct exponents the terms of the sum are

$$\begin{aligned}
&\frac{1}{(k+1+s_1+\epsilon)^2(k+1+s_2+\epsilon)(k+1+s_3+\epsilon)} \\
&\quad = \frac{1}{(s_2-s_1)(s_3-s_1)}\left(\frac{1}{k+1+s_1+\epsilon} - \frac{1}{k+1+s_2+\epsilon}\right)\left(\frac{1}{k+1+s_1+\epsilon} - \frac{1}{k+1+s_3+\epsilon}\right) \\
&\quad \frac{1}{(s_2-s_1)(s_3-s_1)}\left(\frac{1}{(k+1+s_1+\epsilon)(k+1+s_1+\epsilon)} - \frac{1}{(k+1+s_2+\epsilon)(k+1+s_3+\epsilon)}\right. \\
&\qquad \left. + \frac{1}{(k+1+s_1+\epsilon)(k+1+s_3+\epsilon)} - \frac{1}{(k+1+s_2+\epsilon)(k+1+s_3+\epsilon)}\right)
\end{aligned}$$

All but the first term can be further decomposed yielding collapsing sums as in Case 2 above. The first term will yield a rational multiple of $\zeta(3)$ with a denominator with the desired properties.

Case 4: Two distinct exponents appearing twice each. Of the terms

$$\begin{aligned}&\frac{1}{(k+1+s_1+\epsilon)^2(k+1+s_2+\epsilon)^2}\\ &\quad=\frac{1}{(s_2-s_1)^2}\left(\frac{1}{(k+1+s_1+\epsilon)}-\frac{1}{(k+1+s_2+\epsilon)}\right)^2\\ &\quad=\frac{1}{(s_2-s_1)^2}\left(\frac{1}{(k+1+s_1+\epsilon)^2}-\frac{2}{(k+1+s_1+\epsilon)(k+1+s_2+\epsilon)}+\frac{1}{(k+1+s_2+\epsilon)^2}\right)\end{aligned}$$

the middle one will yield a collapsing sum after partial fraction decomposition, and the quadratic terms will yield rational multiples of $\zeta(3)$.

Case 5: One exponent appears three times. We have

$$\begin{aligned}&\frac{1}{(k+1+s_1+\epsilon)^3(k+1+s_2+\epsilon)}\\ &\quad=\frac{1}{(s_2-s_1)}\left(\frac{1}{(k+1+s_1+\epsilon)}-\frac{1}{(k+1+s_2+\epsilon)}\right)\frac{1}{(k+1+s_1+\epsilon)^2}\\ &\quad=\frac{1}{(s_2-s_1)}\left(\frac{1}{(k+1+s_1+\epsilon)^3}-\frac{1}{(k+1+s_1+\epsilon)(k+1+s_2+\epsilon)}\right)\end{aligned}$$

The first term will give rise to a rational multiple of $\zeta(4)$, and the second one will yield a $\zeta(3)$ term and a rational number after partial fraction decomposition, which completes the proof of the lemma.

Lemma 5 *There exists integers A_N, B_N, C_N, and D_N such that*

$$\begin{aligned}0&<A_N\zeta(5)+B_N\zeta(4)+C_N\zeta(3)+D_N\\ &=d_N^5\int_{(0,1)^4}\frac{-\log xyuv(1-u)^N(1-v)^N p_N(x)p_N(y)}{1-xyuv}dxdydudv\\ &<K3^{-N},\end{aligned}$$

for some positive constant K

Proof: The proof will be rather sketchy as it involves much of the same steps as the proof in the previous section. After eliminating the logarithm using a similar integral as in the previous section and going through the sequence of integration by parts in x followed by substitution, followed by integration by parts in y the integral in the Lemma becomes

$$\int_{(0,1)^5}\left(\frac{x(1-x)y(1-y)u(1-u)v(1-v)}{(1-(1-xyuv)w)}\right)^N\frac{dxdydudvdw}{(1-(1-xyuv)w)}.$$

Thus the expression is positive. To estimate the integrant observe that

$$\frac{x(1-x)y(1-y)}{(1-(1-xyuv)w)}\leq\frac{x(1-x)y(1-y)}{(1-(1-xy)w)}<\frac{2}{(1+\sqrt{2})^4}$$

by Beukers' argument. Furthermore,

$$u(1-u)v(1-v) \leq \frac{1}{16}.$$

Therefore the integral can be esimated as follows:

$$\begin{aligned}
\int_{(0,1)^5} &\left(\frac{x(1-x)y(1-y)u(1-u)v(1-v)}{(1-(1-xyuv)w)}\right)^N \frac{dxdydudvdw}{(1-(1-xyuv)w)} \\
&< \frac{2}{(16(1+\sqrt{2})^4))^N} \int_{(0,1)^5} \frac{dxdydudvdw}{(1-(1-xyuv)w)} \\
&= \frac{2}{(16(1+\sqrt{2})^4))^N} \zeta(5)
\end{aligned}$$

Observe that

$$16(1+\sqrt{2})^4 > 543 > 3 \times 173 > 3 \times 2.8^5 > 3d_N^5$$

which yields the desired result.

4 Lessons learned from the repeated exercise in futility

While both approaches ultimately fail there are some valuable lessons from each one. First, it was positive surprise that the sequence of operations involving the transformations of integrals to show positivity is very robust and can handle more complex integrands without deteriorating into an incredible mess. This leaves hope for different scenarios. Secondly, it appears that the quality of the estimate of the integral improves rapidly if the number of variables increases. However, the increase of numbers of variables needs to be paid for by the appearance of undesirable terms involving other values of ζ. On could try to use an approach based on the integral

$$-\int_{(0,1)^3} \frac{(\log x_1x_2x_3)^2}{1-x_1x_2x_3} dx_1dx_2dx_3$$

This would be a compromise of the two approaches presented above. However, since there would be terms in which two of the exponents are equal one will certainly pick up a term that involves $\zeta(3)$. And in the estimate one would get that the integral is bounded by

$$\frac{K}{(4(1+sqrt2)^4)^N}$$

which doesn't outway the factor d_N^5. One glimmer of hope is that the coefficients that appear in the linear combinations of ζ values in Proposition 2 can be explicitely computed and have resemblences with the Apery numbers that appear in the argument for $\zeta(3)$. Using several such linear combinations one may be able to eliminate the undesireable values of ζ while preserving both the positivity and a rapidly decreasing upper bound. But that looks like an incredible mess to begin with.

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