Math 462; Assignment 5

1. Consider $P_1(\mathbb{C})$ and the linear operator T defined by

$$T(a+bz) = b - a + (b+a)z$$

Follow the example in class to find a basis of $P_1(\mathbb{C})$ in which this operator is represented by an upper triangular matrix. I.e. chose a non-zero element v in $P_1(\mathbb{C})$, compute Tv and T^2v and find numbers a_0, a_1, a_2 such that $a_0\mathbf{v} + a_1T\mathbf{v} + a_2T^2\mathbf{v} = 0$. Factor the resulting polynomial and find an eigenvector and an eigenvalue. Then extend this eigen vector to a basis of $P_1(\mathbb{C})$.

Solution: Choose $\mathbf{v} = 1+z$. Then $T(1+z) = 2z$, $T^2(1+z) = 2+2z$. It follows that

$$2\mathbf{v} - T^2\mathbf{v} = 0.$$

Hence, we study the polynomial $p(\xi) = \xi^2 - 2 = (\xi - \sqrt{2})(\xi + \sqrt{2})$

$$\left(T - \sqrt{2}I\right)\left(T + \sqrt{2}I\right)(1+z) = \left(T - \sqrt{2}I\right)(2z + \sqrt{2}(1+z)) = \left(T - \sqrt{2}I\right)(\sqrt{2} + (2+\sqrt{2})z) = 0$$

Therefore $\sqrt{2}$ is an eigenvalue of T, and $\sqrt{2} + (2+\sqrt{2})z$ the corresponding eigenvector. To continue we need to find a vector which is linearly independent of $\sqrt{2} + (2+\sqrt{2})z$. Clearly z satisfies this and $(\sqrt{2} + (2+\sqrt{2})z, z)$ form a basis of $P_1(\mathbb{C})$. To find the matrix (which you weren't asked to do), we compute

$$T(\sqrt{2} + (2+\sqrt{2})z) = \sqrt{2}(\sqrt{2} + (2+\sqrt{2})z),$$

and

$$Tz = 1 + z = \frac{1}{\sqrt{2}}(\sqrt{2} + (2+\sqrt{2})z) - \sqrt{2}z.$$

So $a_{11} = \sqrt{2}$, $a_{12} = \frac{1}{\sqrt{2}}$, and $a_{22} = -\sqrt{2}$.

$$\mathcal{M}(T, ((\sqrt{2} + (2+\sqrt{2})z, z)) = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & -\sqrt{2} \end{pmatrix}$$

2. Consider $P_2(\mathbb{C})$ and let T be the linear operator that represents differentiation. Show that the only invariant subspaces of $P_2(\mathbb{C})$ under T are $\{0\}$ and $P_2(\mathbb{C})$.

Solution: The statement is false and this problem will not be graded.

3. Suppose m and n are positive integers with $m \leq n$. Prove that there exists a polynomial $p \in P_n(F)$ with exactly m distinct roots (i.e. without counting multiplicity). Suppose that $z_1, \ldots, z_{m+1}$ are distinct elements of F and that $w_1, \ldots, w_{m+1} \in F$. Prove that there exists a unique polynomial $p \in P_m(F)$ such that

$$p(z_j) = w_j,$$

for $j = 1, \ldots, m+1$.

Solution: For the first assertion let $z_1, \ldots, z_m$ be m distinct elements od F. Let

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_m)$$

which clearly sis a polynomial with the desired properties.

For the second assertion, let p_k be a polynomial such that $p_k(z_j) = 0$ whenever $j \neq k$ and $p_k(z_k) \neq 0$. Such a polynomial exists for each k by the first assertion. Next let

$$p(z) = \sum_{k=1}^{m+1} \frac{p_k(z)}{p_k(z_k)} w_k$$

If $j \neq k$ we have that

$$\frac{p_j(z_k)}{p_j(z_j)} w_j = 0$$

and for $j = k$

$$\frac{p_k(z_k)}{p_k(z_k)} w_k = w_k$$

therefore

$$p(z_k) = w_k.$$