ZAMM · Z. angew. Math. Mech. **00** (1999) 0, 1–10

WERNER HORN; CAROL ANN SHUBIN

Analytic Treatment of a Bridged Crack Problem

The analysis of the mechanics of a partially bridged crack leads to studying a strongly singularly perturbed integral equation. Rigorous methods are employed to obtain a solution and give its principal asymptotic term. This is done by using an adaption of the Wiener-Hopf factorization to this problem which requires the use of pseudo-differential calculus.

Keywords: Integral Equation, Singular Perturbations

MSC (1991): 45E10

1 Introduction

In this note we examine a partially or completely bridged crack in R^2 which is located along the x_1 axis. We assume that the crack is symmetric about $x_1 = 0$ with end points at $x_1 = \pm c$. Its opening is described by a function $U(x_1)$. The crack is partially or completely bridged by fibers which are perpendicular to it (i.e. in x_2-direction). We assume a linear relationship between the opening of the crack $U(x_1)$ and the bridging force $P(x_1)$, i.e. the force in x_2-direction located at x_1.

$$P(x_1) = g(x_1)U(x_1), \quad \text{with} \quad g(x_1) > 0.$$

The balance of forces acting on the crack is given by

$$T(x_1) = \sigma(x_1) + P(x_1), \qquad -c < x_1 < c,$$

where $T(x_1)$ denote the outside forces and σ denotes the $2, 2$ component of the of the stress tensor. σ satisfies

$$\sigma = C\mathcal{P}\int_{-c}^{c} \frac{b(y_1)}{x_1 - y_1}\, dy_1,$$

where we understand this integral to be the Cauchy principal value. b denotes the dislocation density along the crack, and C is a constant depending on the properties of the material. This equation is obtained by solving the basic field equations for a transversely isotropic linear elastic solid in plane stress or plane strain. This analysis is carried out in more generality in [8]. The crack-opening displacement $U(x_1)$ is related to the dislocation density b via

$$U(x_1) = -\int_{-c}^{x_1} b(y_1)\, dy_1, \qquad U(\pm c) = 0.$$

We combine these quantities and get

$$C\mathcal{P}\int_{-c}^{c} \frac{b(y_1)}{x_1 - y_1}\, dy_1 - g(x_1)\int_{-c}^{x_1} b(y_1)\, dy_1 = T(x_1)$$

To continue we use the following normalizations:

$$x = x_1/c, \qquad u(x) = U(x_1)/c, \qquad \hat{b}(x) = b(x_1)$$

$$\varepsilon = C/c, \qquad \lambda(x) = g(x_1)/C, \qquad -cf(x) = T(x_1)/C.$$

We arrive at

$$\varepsilon \mathcal{P} \int_{-1}^{1} \frac{\hat{b}(y)}{x-y}\, dy - \lambda \int_{-1}^{x} \hat{b}(y)\, dy = -f(x)$$

We substitute $\hat{b}(x) = -u'(x)$ to get the following singularly perturbed integral equation.

$$\lambda(x)u(x) - \varepsilon \frac{d}{dx} \mathcal{P} \int_{-1}^{1} \frac{u(y)}{x-y}\, dy = f(x), \qquad -1 < x < 1, \tag{1}$$

Recall that the operator

$$\mathcal{H}u(x) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\, dy$$

is a bounded linear map $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $1 < p < \infty$ [9]. Moreover, $u \in C^\infty(\mathbb{R})$ implies that $\mathcal{H}u \in C^\infty(\mathbb{R})$. Equation (1) is to be understood in this sense. To abbreviate notation we will drop the $\mathcal{P}$ in the following treatment, with the understanding that singular integrals are Cauchy principal values.

Equation (1) is an example of a strongly singular integral equation. This formulation of the bridged crack problem can be found in [4, 5]. Solutions to this equations were formally computed in [3, 5] and appear in a somewhat different context in [6]. [3] used Wiener-Hopf techniques to analyze the problem. The Wiener-Hopf factorization procedure is appropriate for solving constant coefficient elliptic boundary value problems. This method was generalized by Eskin, see [1], to solve variable coefficient elliptic boundary value problems using pseudo-differential calculus. Indeed, [1] Chapter 27 gives a general treatment of singularly perturbed elliptic pseudo-differential boundary value problems on manifolds. [1] is formulated in much greater generality than needed to solve (1); it's a little like shooting a fly with an elephant gun. This note provides a largely self-contained presentation of the relevant parts of [1]. It also contains some new material: our factorization is appropriate for the interval [8], i.e. we have a simplified microlocalization procedure for studying the boundary behavior of the solution on an interval in one-dimension. Following [1], we will give a rigorous solution to this problem as well as establish the leading order behavior with error estimates in $L^2([-1,1])$.

In the next section we will translate this problem into the setting of pseudo-differential calculus and we will state a result about the existence of solutions and their asymptotic expansion. The solution displays a boundary layer or region of rapid change near the endpoints of the interval [-1,1]. In the third section we will derive this result using following [1]. The advantage of this method is that it is completely algorithmic and can be easily extended to other linear situations. A more general treatment of singularly perturbed integral equations is forthcoming. We believe that these methods will be useful to solving a variety of problems in mechanics that can be reduced to singularly perturbed elliptic boundary value problems.

This investigation was motivated by [7] and is dedicated to the memory of the late C. G. Lange.

2 Notations and statement of the main results

In this section we will reformulate (1) using the notations of pseudo-differential operators. Let $\hat{v}$ denote the usual Fourier transform of v:

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} v(x)e^{i\xi x}\, dx.$$

A pseudo-differential operator $A(x, D)$, where $D = -i\frac{d}{dx}$ is defined by

$$A(x, D)v(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(x,\xi)\hat{v}(\xi)e^{-ix\xi}\, d\xi.$$

The function $A(x, \xi)$ is called the symbol of the pseudo-differential operator. To begin let $l\lambda > 0$ and lf be extensions of λ and f in $L^2(\mathbb{R})$ and consider

$$l\lambda(x)u(x) - \varepsilon\frac{d}{dx}\int_{-\infty}^{\infty}\frac{u(y)}{x-y}\, dy = lf(x). \tag{2}$$

Taking the Fourier transform of

$$-\varepsilon\frac{d}{dx}\int_{-\infty}^{\infty}\frac{u(y)}{x-y}\, dy,$$

we obtain

$$\varepsilon\sqrt{\frac{\pi}{2}}\,|\xi|\,\hat{u}(\xi).$$

To simplify notation let $\epsilon = \varepsilon\sqrt{\frac{\pi}{2}}$ The equation (2) can therefore be rewritten as a pseudo-differential equation

$$A(x, D, \epsilon)u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(x,\xi,\epsilon)\hat{u}(\xi)e^{-ix\xi}\, d\xi = lf(x), \tag{3}$$

with symbol

$$A(x,\xi,\epsilon) = l\lambda(x) + \epsilon\,|\xi|\,.$$

Observe that

$$A(x,\xi,\epsilon) > 0, \qquad \forall\xi\in\mathbb{R}.$$

In particular this means that the equation

$$A(x,\xi,\epsilon) = 0$$

has no real solutions and the corresponding pseudo-differential operator is called elliptic. The main result of this paper concerns the existence of solutions to (1) and the leading terms of their asymptotic behavior.

Theorem 1. *Let $f \in L^2(-1,1)$ and λ be a positive continuous function on [-1,1]. Then (1) has a unique solution u and $u \in L^2(-1,1)$.*

Moreover, if f and λ are smooth functions we have the following principal term of the solution u_ϵ:

$$u_\epsilon(x) = \begin{cases} \left(\frac{1+x}{\varepsilon}\right)^{\frac{1}{2}} \frac{f(x)}{\lambda(x)} & \text{as} \quad \frac{1+x}{\epsilon} \to 0^+ \\ \frac{f(x)}{\lambda(x)} & \text{as} \quad \epsilon \to 0 \\ \left(\frac{1-x}{\varepsilon}\right)^{\frac{1}{2}} \frac{f(x)}{\lambda(x)} & \text{as} \quad \frac{1-x}{\epsilon} \to 0^+ \end{cases} . \tag{4}$$

Remark: Note that this result is independent of the particular extensions of f and λ.

3 Proof of the Theorem

The equation is a variable coefficient singular Wiener-Hopf-type equation. Constant coefficient equations of this type are commonly solved using the Wiener-Hopf factorization method. In [1] this procedure is extended to symbols with non-constant coefficients. This requires the use pseudo-differential calculus. Our proof follows this approach.

Step 1: The operator (3) is strongly elliptic, i. e. it satisfies

$$\int_{-\infty}^{\infty} A(x,\xi,\epsilon)\hat{u}(\xi)\overline{\hat{u}(\xi)}\, d\xi \geq C \left\|u\right\|^2 ,$$

for a suitable positive constant C. In this equation $\overline{\hat{u}}$denotes the usual complex conjugate and $\|\cdot\|$ denotes the L^2-norm. It is well known that strongly elliptic equations admit unique solutions (see, e. g. [1]). Moreover, the solution satisfies:

$$\|u\| \leq C \|f\| .$$

Indeed, this estimate is not sharp, but follows from

$$\|u\|_{H^{1/2}} \leq \|f\|_{H^{-1/2}} ,$$

since in our case $f \in L^2$. (cf. [1]page 207).

Step 2: Let

$$A_0(x,\xi) = l\lambda(x),$$

be the unperturbed (reduced) symbol, i. e. the symbol of (2) with $\epsilon = 0$. The reduced equation

$$A_0(x,\xi)u_0(x) = l\lambda(x)u_0(x) = lf(x)$$

has a unique solution $u_0 \in$, in deed $u_0(x) = \frac{lf(x)}{l\lambda(x)}$. In $[-1,1]$ this becomes $u_0(x) = \frac{f(x)}{\lambda(x)}$. We can write

$$A(x,\xi,\epsilon) = A_0(x,\xi)A_1(x,\epsilon\xi),$$

where

$$A_1(x,\epsilon\xi) = 1 + \frac{|\epsilon\xi|}{l\lambda(x)},$$

carries all the ϵ-dependence of A. Moreover, $A_1(x,\epsilon\xi)$ converges to 1 as $\epsilon \to 0$.

Using this normalization, we see that $A(x,\xi,\epsilon)$ satisfies the hypotheses of [1, Theorem 27.1, p. 346.] (with $A_2 = 0$).

Step 3: To obtain the asymptotic expansions we first expand $A_1(x,\epsilon\xi)$. To abbreviate notation we will omit the x-dependency of $l\lambda$ and use $\eta = \epsilon\xi$ Observe that

$$A_1(x,\eta) = \frac{1}{l\lambda}\left(l\lambda + |\eta|\right) = \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2+2l\lambda|\eta|} = \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2}\sqrt{1+\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}} \tag{5}$$

By the arithmetic-geometric means inequality we have

$$0 \le \frac{2l\lambda|\eta|}{l\lambda^2+\eta^2} \le 1,$$

and therefore the series

$$\sqrt{1+\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}} = 1 + \frac{1}{2}\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2} - \frac{1}{8}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^2 + \frac{1}{16}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^3 + \ldots$$

converges for all $\eta \in \mathbb{R}$ and all $l\lambda > 0$. Combining this with (6) we obtain

$$\begin{aligned} A_1(x,\eta) &= \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} \\ &\quad -\frac{1}{8l\lambda}\left(\frac{l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^2\sqrt{l\lambda^2+\eta^2} \\ &\qquad +\frac{1}{16l\lambda}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^3\sqrt{l\lambda^2+\eta^2} + \ldots \end{aligned}$$

We can write

$$\begin{aligned} A_1(x,\eta) &= \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} + O(|\eta|^{-1/2}) \\ &= A_1^p(x,\eta) + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} + O(|\eta|^{-1/2}) \\ &= A_1^p(x,\eta) + Q(x,\eta). \end{aligned}$$

Observe that the term $R(x,\eta)$ is of lower order, which has a smoothing effect when taking the inverse Fourier transform. A_1^p is called the principal part of A_1. The function $A_1^p(x,\eta)$ can be analytically continued to a domain in the complex plane which contains the entire real line. To do this we have to choose a branch cut starting at $\eta = +il\lambda$ which stays in the entirely in the upper half plane, and a second branch cut starting at $\eta = -il\lambda$ which stays entirely in the lower half plane.

Step 4: The principal part $A_1^p(x,\eta)$ as an analytic function of η can be factored into a plus function $A_1^+(x,\eta)$ and a minus function $A_1^-(x,\eta)$ as follows.

$$A_1^p(x,\eta) = \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} = \frac{1}{\sqrt{il\lambda}}\sqrt{\eta+il\lambda}\frac{1}{\sqrt{-il\lambda}}\sqrt{\eta-il\lambda} = A_1^+(x,\eta)A_1^-(x,\eta),$$

where

$$A_1^+(x,\eta) = \frac{1}{\sqrt{il\lambda}}\sqrt{\eta + il\lambda},$$

and

$$A_1^-(x,\eta) = \frac{1}{\sqrt{-il\lambda}}\sqrt{\eta - il\lambda}.$$

This factorization is consistent with the choice of branches above, i.e. the branch cut for A_1^+ starts at $\eta = -il\lambda$ and stays in the lower half plane, and therefore A_1^+ is analytic in the upper half plane, and analogous for A_1^-.

To continue let $0 < \delta < \epsilon$, and let $\alpha(x)$ be a smooth function with $\alpha(x) = 1$ for $-1 < x < -1 + 2\delta$ and $\alpha(x) = 0$ for $1 - 2\delta < x < 1$. Then we get a new factorization for A_1^p via

$$\begin{aligned} A_3^+(x,\eta) &= \left(A_1^+(x,\eta)\right)^{\alpha(x)}\left(A_1^-(x,\eta)\right)^{1-\alpha(x)} \\ A_3^-(x,\eta) &= \left(A_1^-(x,\eta)\right)^{\alpha(x)}\left(A_1^+(x,\eta)\right)^{1-\alpha(x)} \end{aligned}$$

Observe that

$$A_1^p(x,\eta) = A_3^+(x,\eta)A_3^-(x,\eta)$$

for all $x \in [-1,1]$ and that

$$\begin{aligned} A_3^+(-1,\eta) = A_1^+(-1,\eta), &\qquad A_3^-(-1,\eta) = A_1^-(-1,\eta), \\ A_3^+(1,\eta) = A_1^-(1,\eta), &\qquad A_3^-(1,\eta) = A_1^+(1,\eta). \end{aligned}$$

This process is an extension of the process found in [1], as developed in [8].

The purpose of such a factorization is to change the singular perturbation problem, which has the small parameter ϵ in the highest order term, into a regular perturbation problem where ϵ is only in lower order terms.

Now let us consider the extended differential equation

$$A(x,D,\epsilon)u_\epsilon = lf + f_\epsilon^-,$$

where f_ϵ^- is supported outside $[-1,1]$. Following the steps of [1] we write $u_\epsilon = (A_3^+)^{-1}v_\epsilon$ and apply the operator

$$p(A_3^-)^{-1},$$

to the equation. Here p denotes the restriction to $[-1,1]$. We get

$$\begin{aligned} p(A_3^-)^{-1}A(A_3^+)^{-1}v_\epsilon &= p(A_3^-)^{-1}lf \\ p(A_3^-)^{-1}(A_0A_p^1 + A_0Q)(A_3^+)^{-1}v_\epsilon &= p(A_3^-)^{-1}lf \\ p(A_0 + Q_1)v_\epsilon &= p(A_3^-)^{-1}lf \end{aligned}$$

In the last expression Q_1 consists of the term $p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}$ and the lower order terms which arise from the rules of composition for pseudo-differential operators. The principal part of $p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}$ is given by

$$p\left(\frac{|\eta|}{l\lambda^2+\eta^2}-\frac{l\lambda}{4}\left(\frac{|\eta|}{l\lambda^2+\eta^2}\right)^2+\cdots\right).$$

It follows that there exist constants C_s such that

$$\left\|p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}v_\epsilon\right\|_{H^{s-1}(\mathbb{R})}\leq C_s\epsilon\left\|v_\epsilon\right\|_{H^s(\mathbb{R})},\qquad\forall\epsilon>0,\qquad\forall s.$$

In particular this statement holds for $s=1$, which gives an L^2-error bound for u. Theorem 18.3 (composition of pseudo-differential operators) of [1] implies a similar estimate for the remaining parts of Q_1 (cf. [1, p. 348, equation (27.54)].

Following the computations of [1, p. 248] the explicit solution u_ϵ is given by

$$u_\epsilon=(A_3^+)^{-1}(I+R_0pA_1^p)^{-1}R_0p(A_3^-)^{-1}lf, \tag{6}$$

where I is the identity operator, and R_0 is the inverse of A_0. In our specific case R_0 is just multiplication by $1/l\lambda$.

Step 5: We next obtain an explicit expression near $x=-1$, the expression near $x=1$ follows from an analogous computation. Near $x=-1$, we can use A_1^- and A_1^+ directly. For sufficiently smooth functions f, (6) can be rewritten as

$$u_\epsilon=(A_3^+)^{-1}v_\epsilon+u_\epsilon^{(0)}, \tag{7}$$

where v_ϵ is the solution to

$$pA_0v_\epsilon=p(A_3^-)^{-1}lf,$$

and $u_\epsilon^{(0)}$ is smooth. Following the steps in [1] we see that

$$v_\epsilon=u_0+u_\epsilon^{(1)},$$

where $u_0=f/l\lambda$, the solution to the reduced equation, and $u_\epsilon^{(1)}$ is smooth. I. e. we get

$$u_\epsilon=(A_3^+)^{-1}u_0+u_\epsilon^{(3)}. \tag{8}$$

Next observe that near $x=-1$ we, can localize the symbol for $(A_3+)^{-1}$ as follows:

$$\begin{aligned}A_3^+(x,\eta)^{-1}&=\frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}}+B_+(x,\eta),\\&=A_1^+(-1,\eta)^{-1}+B_+(x,\eta)\end{aligned}$$

where B_+ is a lower order term. Computing the inverse Fourier transform of $A_1^+(-1,\eta)^{-1}$ we get

$$\begin{aligned}a(\frac{x}{\epsilon})&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}}e^{ix\xi}\,d\xi\\&=\frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}\frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}}e^{i\eta\frac{x}{\epsilon}}\,d\eta\\&=\frac{1}{\sqrt{\pi}\epsilon}\left(\frac{x}{\epsilon}\right)^{-\frac{1}{2}}e^{-\lambda(-1)\frac{x}{\epsilon}}\Theta(x)\end{aligned}$$

where $\Theta(x)$ denotes the usual Heaviside function. Observe that $a(x/\epsilon) \to \delta(x)$ as $\epsilon \to 0$. On the other hand we may represent the leading term in the pseudo-differential equation (8) as a convolution.

$$u_\epsilon(y) = \int_{-\infty}^{\infty} a(\frac{x}{\epsilon}) u_0(y-x)\, dx + u_\epsilon^{(4)} \tag{9}$$

Observe that $a(\frac{x}{\epsilon}) = 0$ for $x < 0$ and that $u_0(y-x) = 0$ for $y - x < -1$, i.e. for $x > 1 + y$. Thus we can replace the limits of integration in (9) to get

$$u_\epsilon(y) = \int_0^{1+y} a(\frac{x}{\epsilon}) u_0(y-x)\, dx + u_\epsilon^{(4)}.$$

Finally, we expand $u_0(y-x)$ into a Taylor series around y to get

$$u(y-x) = u(y) + u'(y)x + \ldots$$

Omitting the higher order terms we obtain

$$\begin{aligned} u_\epsilon(y) &= \int_0^{1+y} a(\frac{x}{\epsilon}) u_0(y)\, dx + u_\epsilon^{(5)} \\ &= \int_0^{1+y} \frac{1}{\sqrt{\pi}\epsilon} \left(\frac{x}{\epsilon}\right)^{-\frac{1}{2}} e^{-\lambda(-1)\frac{x}{\epsilon}} u_0(y)\, dx + u_\epsilon^{(5)} \\ &= \int_0^{1+y} \frac{1}{\epsilon} \left(\frac{x}{\epsilon}\right)^{-\frac{1}{2}} u_0(y)\, dx + u_\epsilon^{(6)} \\ &= u_0(y) \left(\frac{1+y}{\epsilon}\right)^{\frac{1}{2}} + u_\epsilon^{(6)} \end{aligned}$$

In this computation we used the fact that $e^{-\lambda(-1)\frac{x}{\epsilon}}$ is rapidly decreasing for $\frac{x}{\epsilon} > 0$ with a leading term of 1.

Graphs of the term

$$S(y,\epsilon) = \int_0^{1+y} \frac{1}{\epsilon} \left(\frac{x}{\epsilon}\right)^{-\frac{1}{2}} e^{-\lambda(-1)\frac{x}{\epsilon}}\, dx$$

are shown in Figure 1 and 2. In both cases we used $\lambda(-1) = 1$. In Figure 1 we used $\epsilon = 1$ in Figure 2 we used $\epsilon = 0.1$. One can see that this function approaches a step function as $\epsilon \to 0$ in a rather rapid way.

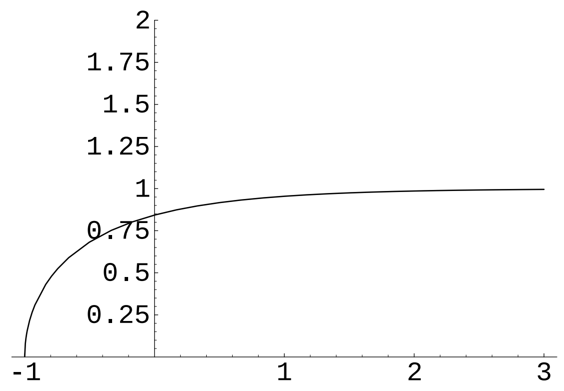

Figure 1: The graph of $S(y,\epsilon)$ for $\epsilon = 1$.

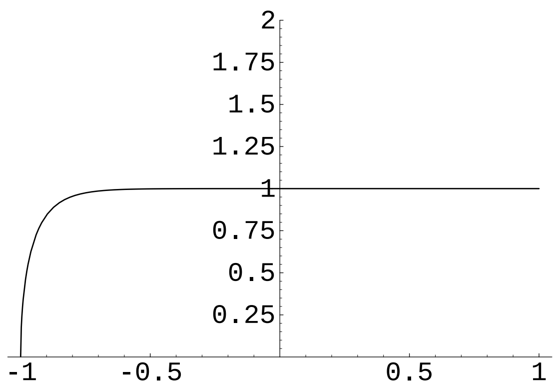

Figure 2: The graph of $S(y,\epsilon)$ for $\epsilon = 0.1$.

The leading terms of the asymptotic expansion near $x = 1$ are computed in an analogous way. The only difference is that we use $A_1^-(+1,\eta)$. For x in the interior of $(-1,1)$, u_ϵ converges pointwise to u_0 as $\epsilon \to 0$. This completes the proof of the theorem.

4 Concluding Remarks

There are other ways to obtain the leading order terms for the solution to (1). Indeed we can explicitly compute the inverse Fourier transform of $(A_1^p(x,\eta))^{(-1)} = \frac{\lambda}{\sqrt{\lambda^2+\eta^2}}$ as follows:

$$
\begin{aligned}
\tilde{a}(x/\epsilon) &= \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{\lambda}{\sqrt{\lambda^2+\eta^2}} e^{-ix\xi}\, d\xi \\
&= \frac{1}{2\pi\epsilon}\lambda(x)\int_{-\infty}^{\infty} \frac{e^{-i\frac{x}{\epsilon}\eta}}{\sqrt{\lambda^2+\eta^2}}\, d\eta \\
&= \frac{\lambda(x)}{\epsilon\pi} K_0(\left|\frac{x}{\epsilon}\right| \lambda(x)).
\end{aligned}
$$

Here $K_0(\cdot)$ denotes the modified Bessel function (see, for example, [2], for the definition of this function).

As in the previous section can write the solution as a convolution. First observe that $u_0(y-x) = 0$ except when $-1 \le x - y \le 1$. Then write $u_0(y-x) = u_0(y) - u_0'(y)x + \ldots$. Using the leading term we get

$$
u_\epsilon(y) = u_0(y)\int_{y-1}^{y+1} \frac{\lambda(x)}{\epsilon\pi} K_0(\left|\frac{x}{\epsilon}\right| \lambda(x))\, dx = u_0(y)S(y,\epsilon).
$$

Then we make a change of variables in the integral and let $s = x/\epsilon$ to arrive at

$$
\begin{aligned}
S(y,\epsilon) &= \int_{\frac{y-1}{\epsilon}}^{\frac{y+1}{\epsilon}} \frac{\lambda(\epsilon s)}{\pi} K_0(|s|\,\lambda(\epsilon s))\, ds \\
&= \int_{\frac{y-1}{\epsilon}}^{\frac{y+1}{\epsilon}} \frac{\lambda(0)}{\pi} K_0(|s|\,\lambda(0))\, ds + O(\epsilon) = \tilde{S}(y,\epsilon) + O(\epsilon).
\end{aligned}
$$

Using the properties of the modified Bessel function $\tilde{S}(y,\epsilon)$ can be explicitly computed and its graph is shown in Figures 3 and 4. This method appears to be easier and more straight forward. However, the method does not

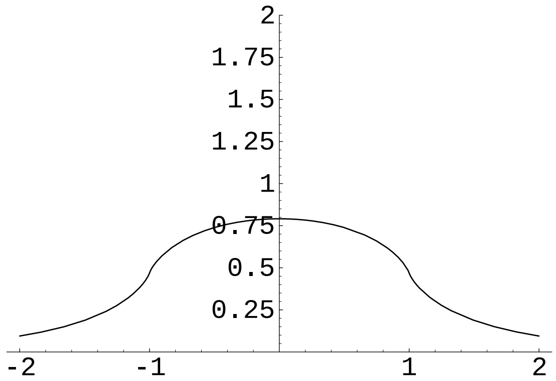

Figure 3: The graph of $\tilde{S}(y,\epsilon)$ for $\epsilon = 1$.

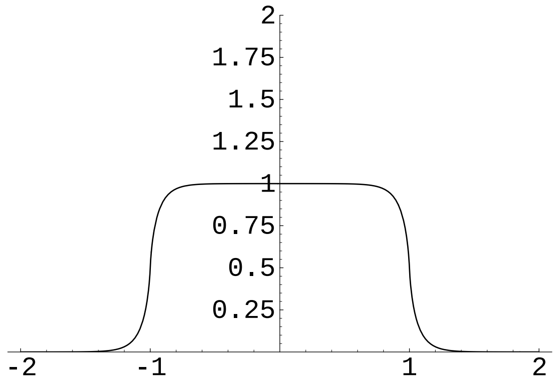

Figure 4: The graph of $\tilde{S}(y,\epsilon)$ for $\epsilon = 0.1$.

guarantee that the solution is supported in $[-1,1]$, but only that the solution will decrease rapidly outside this interval. This makes the solutions depend on the specific extensions lf of f and $l\lambda$ of λ. The factoring of $A_1^p(x,\eta)$ into a $+$ and a $-$ symbol as done in in the previous section will guarantee that the support of the solution is contained in $[-1,1]$, and that the leading terms of the asymptotics will be independent of the extension of the solution. The method of this section will yield good results in the interior of $[-1,1]$, but it is not very useful near the endpoints.

Acknowledgement. Professor Shubin was partially supported by NSF Grant NSF-DMS-9623518. We would also like to express our gratitude to Professor G. I. Eskin of UCLA who was the thesis advisor for Professor Shubin.

References

1 G. I. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations,* American Mathematical Society, Providence, R. I., 1981

2 C. M. Bender, S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* McGraw-Hill Book Company, New York, 1978.

3 A. K. Gautesen, W. E. Olmstead, *Asymptotic solution of some singularly perturbed Fredholm integral equations,* J. Appl. Math. Phys. (ZAMP), **40** (1989), 230–244.

4 M. Hori, S. Nemat-Nasser, *Toughening by partial or full bridging of cracks in ceramics and fiber reinforced composites,* Mech. of Materials, **6** (1987), 245–269.

5 M. Hori, S. Nemat-Nasser, *Asymptotic solution of a class of strongly singular integral equations,* SIAM J. Appl. Math., **50(3)** (1990), 716–725.

6 W. T. Koiter, *On the diffusion load from a stiffener into a sheet,* Quart. J. Mech. Appl. Math., **8** (1955), 164–178.

7 C. G. Lange, D. R. Smith, *Singular perturbation analysis of integral equations,* Studies in Appl. Math., **79** (1988), 1–63.

8 C. A. Shubin, *Dissertation*, Adviser: G. I. Eskin, UCLA, 1989, (unpublished).

9 E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton University Press, Princeton, 1970.

Address: Werner Horn; Carol Ann Shubin, California State University Northridge, Northridge, CA 91330-8313, USA, email: `werner.horn, carol.shubin@csun.edu`