Homework solutions

1. Prove the last proposition.

   Solution: Let $a_n < b_n$ and $L = \limsup_{n\to\infty} a_n$ and $M = \limsup_{n\to\infty} b_n$. Then fior any $\epsilon > 0$ there is a number N such that $b_n < M + \epsilon$ for all $n \geq N$. It follows that $a_n < b_n < M + \epsilon$ for all $n \geq N$. Assume that $L > M$ and let $\epsilon = (M - L)/3$. we have

   $$a_n < b_n < M + \epsilon < L - \epsilon$$

   for all $n \geq N$. Therefore the interval $(L - \epsilon, L + \epsilon)$ contains only finitely many terms of the sequence and L cannot be the $\limsup a_n$. The proof for the $\liminf$ follows the same pattern.

2. Let $\{a_n\}$ and $\{b_n\}$ be to bounded sequences. Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

   Solution: Let $L = \limsup a_n$, and $M = \limsup b_n$ then for any $\epsilon > 0$ there exists a number N such that

   $$a_n < L + \frac{\epsilon}{2}, \qquad \text{and} \qquad b_n < M + \frac{\epsilon}{2},$$

   for all $n \geq N$. Adding these two inequalities yields

   $$a_n + b_n < L + M + \epsilon,$$

   for all $n \geq N$. In this inequality we have two sequences – on the left the sequence $a_n + b_n$ and on the right a constant sequence. Problem 1 implies that

   $$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(L + M + \epsilon) = L + M + \epsilon,$$

   for all $\epsilon > 0$, and thus $\limsup_{n\to\infty}(a_n + b_n) \leq L + M$.

3. Find an example of sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n < b_n$ for all n, but $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n$.

   Solution: Let $a_n = \frac{n-1}{n}$ and $b_n = 1$. Then $a_n < b_n$ for all $n \in \mathbb{N}$ but

   $$\limsup_{n\to\infty} a_n = 1 = \limsup_{n\to\infty} b_n.$$

4. Let $a_n = \sin n$. Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. Moreover, show that the sequence has a subsequence with subsequential limit point $\frac{1}{2}$
   Solutions: to be provided later.