KINETIC THEORY OF ATOMS AND PHOTONS: ON THE SOLUTION TO THE MILNE–CHANDRASEKHAR PROBLEM

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Mathematics Subject Classification. 76N15, 82C40, 82C70

Keywords. Gas–radiation interaction, Transport processes, Approximated and numerical solutions

ABSTRACT.

A physical system of monochromatic photons interacting with gas–particles endowed with two internal energy levels is considered. A nonlinear transport problem for photons (the so–called Milne–Chandrasekhar problem) is then formulated, treating separately the stationary and the time–dependent case. For the former, an approximate analytical solution as well as existence and uniqueness results are obtained, while for the latter, the numerical results are provided together with their comparisons with those obtained by Chandrasekhar himself.

1. INTRODUCTION

A continuum approach to the dynamics of a gas which interacts with a radiation field is a quite established field in both Aerospace Engineering and Astrophysics [1–3]. A kinetic theory of atoms interacting with photons is a more recent field of interest [4,5], especially in view of the applications in Astrophysics and Plasma Physics.

In a recent paper [6], Rossani, Spiga and Monaco have proposed a kinetic model for the study of a physical system constituted by two–level atoms and monochromatic photons. Such a model incorporates the basic feature that allows a good description of the system,

namely the interplay of inelastic collisions between atoms, on one side, and interaction between gas and radiation, on the other. An interesting application to test reliability of the kinetic model is the Milne–Chandrasekhar problem [7,8].
Such a problem has been treated in paper [9], using the moment equations derived by the kinetic model proposed in [6]. The problem itself results to be an initial-boundary value problem for a set of three nonlinear PDEs. In paper [9] existence and uniqueness theorems for its solution have been proved.
The present work is the second part of [9] and its principal aim consists in solving the problem under consideration by approximate analytical methods and numerical techniques. We consider separately the stationary and the time-dependent problem. For the stationary problem we provide existence and uniqueness results and then we derive an approximate explicit solution for the same problem by resorting to the Eddington approximation [2]. Next, we provide numerical solutions to the transient (time-dependent) problem. Finally, a comparison between the results of the present model and those obtained by a simplified one (basically the one adopted in the past by Chandrasekhar, who has not included inelastic interactions among atoms) is given for various physical situations.

2. FORMULATION OF THE PROBLEM

The following physical system is considered:

- A gas of atoms A of mass m endowed with only two internal energy levels E_1 and E_2, $E_1 < E_2$. In what follows we will denote by A_1 and A_2, respectively, particles A at levels "1" and "2" (fundamental and excited levels).
- A radiation field of photons p interacting with gas–particles at a fixed frequency $\nu = \Delta E/h$, $\Delta E = E_2 - E_1$, h being the Planck constant.

The physical conditions of the system are characterized by the following inequalities

$$k_B T << mc^2, \qquad \Delta E << c\sqrt{8mk_BT/\pi} \tag{2.1}$$

where T is the temperature of the gas, k_B the Boltzmann constant and c the speed of light. The first inequality assures that relativistic effects can be neglected, while the second guarantees that the photon momentum is much smaller than the mean thermal one of the gas, so that exchange of momentum during the interactions between photons and atoms can be neglected as well. In such physical conditions [2,4] only the following interactions between particles and between particles an photons take place:

a) Elastic interactions among particles with the same or different energy level.
b) Inelastic mechanism of exchange between the internal energy levels

$$A_1 + A_1 \rightleftharpoons A_2 + A_1, \qquad A_1 + A_2 \rightleftharpoons A_2 + A_2.$$

c) Interactions between gas–particles and photons
   c.1) Absorption: $\quad A_1 + p \to A_2$

c.2) Spontaneous emission: $A_2 \to A_1 + p$

c.3) Stimulated emission: $A_2 + p \to A_1 + 2p$.

In paper [6], taking into account the above collisional scheme, the kinetic equations have been derived together with the corresponding moment equations . In paper [9] a suitable closure for such equations has been obtained under the assumptions:

i) the characteristic relaxation time of elastic collision processes is much smaller compared to the ones relevant to inelastic and gas–radiation interaction processes,

ii) the gas is confined to a slab,

iii) the number densities n_1 and n_2 of atoms at energy level E_1 and E_2, respectively, are such that

$$n_1(t,x) + n_2(t,x) = n = const, \qquad \text{for all } (t,x),$$

iv) the gas has zero mean velocity for all (t,x).

According to that closure, hereinafter the physical system under consideration will be ruled by the equations

$$\frac{\partial T}{\partial t} = \frac{2h\nu}{3nk_B}[n_1\gamma_{12}(T) + n_2\gamma_{22}(T)][n_2 - n_1\exp(-h\nu/k_BT)] \tag{2.2}$$

$$\frac{\partial n_2}{\partial t} = [n_1\gamma_{12}(T) + n_2\gamma_{22}(T)][n_1\exp(-h\nu/k_BT) - n_2] + \beta\mathcal{I}(n_1 - n_2) - 4\pi\alpha n_2 \tag{2.3}$$

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial x} = h\nu[\alpha n_2 - \beta I(n_1 - n_2)]. \tag{2.4}$$

In these equations $I = I(t,x,\mu)$ is the radiation intensity, $\mu = \cos\theta$, and θ is the angle between the x-axis and the photon velocities. Moreover α and β are the Einstein's coefficients [1], accounting the emission and absorption rates, whereas γ_{12} and γ_{22} are the inelastic collisional frequencies which turn out to be, in general, functions of the gas temperature $T(t,x)$. Finally $\mathcal{I}$ is the integrated radiation intensity defined by

$$\mathcal{I}(t,x) = 2\pi\int_{-1}^{1} I(t,x,\mu)d\mu.$$

We are now ready to formulate the so–called Milne–Chandrasekhar problem (for origin and motivations of the problem (see sec. 4 of [9]). Consider a slab of width $2a$, $-a \le x \le a$, filled with a gas:

- For $t \in (-\infty, 0)$ the gas at a known number density n is in absolute equilibrium at a temperature T_0.
- The boundaries of the slab are perfectly reflecting mirrors, so that also the radiation field, at temperature T_0, is in equilibrium.
- At $t = 0^+$ the mirrors are removed and the gas in the slab is subjected to zero radiation at the left boundary $x = -a$ and to a known radiation $I^*(\mu)$ at the right one $x = a$.

Accordingly, we set the following initial and boundary conditions to be considered for equations (2.2–2.4):

$$\forall x \in [-a,a]: \qquad \begin{cases} I(0,x,\mu) = \dfrac{\alpha/\beta}{\exp(h\nu/k_B T_0)-1} \qquad \forall \mu \in [-1,1] \\ n_2(0,x) = \dfrac{n}{\exp(h\nu/k_B T_0)+1} \\ T(0,x) = T_0 \end{cases} \tag{2.5}$$

$$\forall t > 0: \qquad \begin{cases} I(t,-a,\mu) = 0 \ , \quad \forall \mu > 0 \\ I(t,a,\mu) = I^*(\mu) \ , \quad \forall \mu < 0 \ . \end{cases} \tag{2.6}$$

The first and second initial datum (2.5) expresses, at $T = T_0$, the equilibrium condition [6] for the radiation intensity (Planck's law of radiation) and for the number density of atoms at the excited energy level.

Remark 1: The model adopted in the past by Chandrasekhar [8] can be obtained from equations (2.2–2.4):

i) by setting $\gamma_{12} = \gamma_{22} = 0$. This implies that $T = T_0$ for all (t,x),

ii) by neglecting $(1/c)(\partial I/\partial t)$ as compared to $\mu(\partial I/\partial x)$,

iii) by approximating $n_1 - n_2$ with n_1, and treating n_1 as a constant.

3. ANALYSIS OF THE STATIONARY STATE

We consider now the stationary solutions of the problem formulated in the previous section. By setting $\partial/\partial t = 0$ in equations (2.2–2.4), we obtain

$$n_2 - n_1 \exp(-h\nu/k_B T) = 0 \tag{3.1}$$

$$[n_1\gamma_{12}(T) + n_2\gamma_{22}(T)][n_1 \exp(-h\nu/k_B T) - n_2] + n_1\beta\mathcal{I} - n_2(4\pi\alpha + \beta\mathcal{I}) = 0 \tag{3.2}$$

$$\mu\frac{\partial I}{\partial x} = h\nu[n_2(\alpha + \beta I) - n_1\beta I] \ . \tag{3.3}$$

After substituting (3.1) in equation (3.2) one obtains

$$n_1\beta\mathcal{I} - n_2(4\pi\alpha + \beta\mathcal{I}) = 0 \ . \tag{3.4}$$

The last equation together with $n_1 + n_2 = n$ gives

$$n_1(x) = n\,\frac{4\pi\alpha + \beta\mathcal{I}(x)}{4\pi\alpha + 2\beta\mathcal{I}(x)}, \tag{3.5}$$

$$n_2(x) = n\,\frac{\beta\mathcal{I}(x)}{4\pi\alpha + 2\beta\mathcal{I}(x)} \ . \tag{3.6}$$

From (3.1) and (3.4) we find

$$T(x) = \frac{h\nu/k_B}{\log\left[1 + 4\pi\alpha/(\beta\mathcal{I}(x))\right]} \ . \tag{3.7}$$

Remark 2: In the stationary case the integrated intensity $\mathcal{I}$ and temperature T are linked, through a local Planckian relationship

$$\mathcal{I}(x) = \frac{4\pi\alpha/\beta}{\exp(h\nu/k_B T(x)) - 1} \ .$$

We want to point out that in the case of Chandrasekhar's simplified model (that does not include inelastic interaction terms) there is not such a linkage between $\mathcal{I}(x)$ and $T(x)$.

By substituting n_1 and n_2 from equations (3.5) and (3.6) into equation (3.3) one gets

$$\mu\frac{\partial I}{\partial x} = \Gamma(\mathcal{I})(\mathcal{I} - 4\pi I) \tag{3.8}$$

where

$$\Gamma(\mathcal{I}) = \frac{h\nu n}{4\pi/\beta + 2\mathcal{I}/\alpha} \ .$$

After integrating (3.8) and taking into account the boundary conditions (2.6), we obtain

$$\begin{aligned} I(x,\mu) =&\Theta(\mu)\int_{-a}^{x}\frac{1}{\mu}\mathcal{I}(x')\Gamma(\mathcal{I}(x'))\exp\left[-\int_{x'}^{x}\frac{4\pi}{\mu}\Gamma(\mathcal{I}(x''))dx''\right]dx' \\ &+\Theta(-\mu)\left(I^*(\mu)\exp\left[\int_{x}^{a}\frac{4\pi}{\mu}\Gamma(\mathcal{I}(x'))dx'\right]\right. \\ &\left.-\int_{x}^{a}\frac{1}{\mu}\mathcal{I}(x')\Gamma(\mathcal{I}(x'))\exp\left[\int_{x}^{x'}\frac{4\pi}{\mu}\Gamma(\mathcal{I}(x''))dx''\right]dx'\right) \ , \end{aligned} \tag{3.9}$$

where Θ is the Heaviside function. Thus, we have

$$\begin{aligned} \mathcal{I}(x) = 2\pi&\int_0^1\exp\left[-\int_x^a\frac{4\pi}{\mu}\Gamma(\mathcal{I}(x'))dx'\right]I^*(-\mu)d\mu \\ &+2\pi\int_{-a}^{a}\mathcal{I}(x')\Gamma(\mathcal{I}(x'))E_1\left(\left|\int_x^{x'}4\pi\Gamma(\mathcal{I}(x''))dx''\right|\right)dx'. \end{aligned} \tag{3.10a}$$

equation (3.10a) is a nonlinear integral equation for $\mathcal{I}(x)$, where E_1 is the exponential integral of first order, defined by

$$E_1(\xi) = \int_0^1\frac{\exp(-\xi/\mu)}{\mu}d\mu. \tag{3.10b}$$

Once equation (3.10a) is solved for $\mathcal{I}(x)$, $I(x,\mu)$ can computed from equation (3.9). Finally, n_1 and n_2 are obtained from equations (3.5) and 3.6), respectively, and T from equation (3.7).

Equation (3.10a) is very complicated and its solution would require suitable *ad hoc* numerical methods. In our search for approximated but analytical solutions, we observe that if I^* not strongly peaked as a function of μ, and since the *gain term* in equation (3.8) is independent of μ, it is reasonable to adopt the so called Eddington approximation for I (see, for example, [2]). Equivalently, we will assume that $I(x,\mu)$ depends linearly on μ. Next, we will show

how within the above approximation, one can obtain the analytical solution of the still nonlinear problem. A posteriori, we will also check whether the present approach is good by comparing the analytical results from the Eddington approximation with those obtained via the numerical integration of equations (2.2–2.4) considered together with the boundary conditions (2.5) and (2.6).

Before starting the analysis the Eddington approximation, it is useful to observe that integration of equation (3.8) with respect to μ gives

$$\frac{dj_{\mathrm{r}}}{dx} = 0\ , \qquad j_{\mathrm{r}}(x) = \int_{-1}^{1} \mu I(x,\mu) d\mu,$$

and also $j_{\mathrm{r}}(x) = j_{\mathrm{r}}(-a)$. This means that $j_{\mathrm{r}}(x)$ is a negative constant, since, due to the boundary condition at $x = -a$, we have $j_{\mathrm{r}}(-a) < 0$. Now, the Eddington approximation means that $I(x,\mu)$ has the form

$$I(x,\mu) = \frac{1}{4\pi}[\mathcal{I}(x) + 3\mu j_{\mathrm{r}}(x)]\ ,$$

and equation (3.8) can be expressed as

$$\frac{\partial I}{\partial x} = 3|j_{\mathrm{r}}|\Gamma(\mathcal{I})\ . \tag{3.11}$$

Next, multiplying equation (3.11) by 2π and integrating it with respect to $d\mu$, we obtain the equation for $\mathcal{I}$

$$\frac{d}{dx}\,\mathcal{I} = 12\pi|j_{\mathrm{r}}|\Gamma(\mathcal{I})\ . \tag{3.12}$$

We still have to formulate suitable boundary conditions for $\mathcal{I}(x)$. It is well known [2] that the exact boundary conditions (2.6) cannot be satisfied within the present approximation. Instead, we use the approximated integral boundary conditions that are derived from the original boundary conditions (2.6). Indeed, after integrating (2.6) with respect to μ, we have

$$\int_0^1 I(-a,\mu)\mu d\mu = 0 \tag{3.13}$$

and

$$\int_{-1}^{0} I(a,\mu)\mu d\mu = \int_{-1}^{0} I^*(\mu)\mu d\mu. \tag{3.14}$$

Conditions (3.13) and (3.14) yield

$$\frac{1}{2}\mathcal{I}(-a) = |j_{\mathrm{r}}| \tag{3.13$'$}$$

and

$$\frac{1}{2}\mathcal{I}(a) + |j_{\mathrm{r}}| = 2\pi I^{\bullet}, \tag{3.14$'$}$$

where

$$I^{\bullet} = 2\int_0^1 I^*(-\mu)\mu d\mu,$$

is a weighted average of $I^*(\mu)$. Now, we can integrate equation (3.12) to obtain

$$\beta\mathcal{I}^2(x) + 4\pi\alpha\mathcal{I}(x) = \mathcal{L}(x) \tag{3.15}$$

where

$$\mathcal{L}(x) = \beta\mathcal{I}^2(-a) + 4\pi\alpha\mathcal{I}(-a) + 12\pi|j_{\mathrm{r}}|h\nu n\alpha\beta(x+a).$$

Solving equation (3.15) for $\mathcal{I}(x)$ we get

$$\mathcal{I}(x) = \frac{\mathcal{L}(x)}{2\pi\alpha + \mathcal{R}(x)}\ , \tag{3.16}$$

with

$$\mathcal{R}(x) = \sqrt{(2\pi\alpha)^2 + \beta\mathcal{L}(x)}\ .$$

The constants $\mathcal{I}(-a)$ and $|j_{\mathrm{r}}|$ can be determined from the boundary conditions (3.13′) and (3.14′). Furthermore,

$$4\pi\alpha[\mathcal{I}(a) - \mathcal{I}(-a)] + \beta[\mathcal{I}^2(a) - \mathcal{I}^2(-a)] = 24\pi|j_{\mathrm{r}}|h\nu n\alpha\beta a\ . \tag{3.15′}$$

The identity (3.15′) is simply (3.15) with $x = a$. Finally, after easy calculations, we get

$$|j_{\mathrm{r}}| = \pi I^\bullet\ \frac{\alpha + \beta I^\bullet}{(3/2)h\nu n\alpha\beta a + \alpha + \beta I^\bullet}$$

and

$$\mathcal{I}(-a) = 2|j_{\mathrm{r}}|\ .$$

Equation (3.16) gives explicit solution of the stationary problem within the Eddington approximation. Also (3.16) and (3.12) imply that $\mathcal{I}(x)$ is positive, monotonically increasing and concave for $x \in [-a,a]$. Moreover, as before, knowing $\mathcal{I}$ we can compute $n_2(x)$ and $T(x)$. They are

$$n_2(x) = \frac{n\beta\mathcal{L}(x)}{4\pi\alpha[2\pi\alpha + \mathcal{R}(x)] + 2\beta\mathcal{L}(x)} \tag{3.17}$$

and

$$T(x) = \frac{h\nu/k_B}{\log\left(\frac{4\pi\alpha}{\beta}\ \frac{2\pi\alpha + \mathcal{R}(x)}{\mathcal{L}(x)} + 1\right)}\ . \tag{3.18}$$

4. EXISTENCE AND UNIQUENESS RESULTS

In this section we prove existence and uniqueness theorems for the system of equations (3.1–3.3) with the boundary conditions as in (2.6). First, we state the existence result

Theorem 4.1. *Suppose that the boundary condition* $0 \le I^* \in L^\infty(-1,0)$. *For any* $a > 0$, $n > 0$, $\alpha > 0$, $\beta > 0$, *and* $\nu > 0$ *the system*

$$n_1 + n_2 = n, \tag{4.1}$$

$$n_2 - n_1 \exp(-h\nu/k_B T) = 0, \tag{4.2}$$

$$[n_1\gamma_{12}(T) + n_2\gamma_{22}(T)][n_1 \exp(-h\nu/k_B T) - n_2] + n_1\beta\mathcal{I} - n_2(4\pi\alpha + \beta\mathcal{I}) = 0, \tag{4.3}$$

$$\mu\frac{\partial I}{\partial x} = h\nu[n_2(\alpha + \beta I) - n_1\beta I] \ , \tag{4.4}$$

$$\begin{aligned} I(t, -a, \mu) &= 0, \qquad \forall \mu > 0, \\ I(t, a, \mu) &= I^*(\mu), \quad \forall \mu < 0 \ , \end{aligned} \tag{4.5}$$

has a solution $(n_1, n_2, T, I) \in L^\infty(-a,a) \times L^\infty(-a,a) \times L^\infty(-a,a) \times L^\infty((-a,a)\times(-1,1))$, *with* $0 \le I(x,\mu) \le \|I^*\|_{L^\infty}$, *almost everywhere, and* $\mu\frac{\partial I}{\partial x} \in L^\infty((-a,a)\times(-1,1))$. *Furthermore, the functions* n_1, n_2, T, *and* $\mathcal{I} = 2\pi\int_{-1}^{1} I\, d\mu$ *are nonnegative and Hölder continuous on* $[-a,a]$.

The uniqueness result that we can prove requires smallness of the dimensionless thickness of the slab. More precisely, we have

Theorem 4.2. *Suppose that the assumptions of Theorem 4.1 are satisfied and* $a\beta n h\nu \equiv a\beta n\Delta E$ *is sufficiently small, then the solution obtained in Theorem 4.1 is unique.*

The proof of Theorem 4.1 is based on the compactness argument already used by the authors in [9]. Furthermore, some of the arguments below are similar to those in the proof of Theorem 1 of [13]. For completeness, we state the main compactness result and give a sketch of the proof with emphasis on independence of an upper bound of I on smallness of $a\beta n\Delta E$. This crucial in the proof of Theorem 4.2. In fact, (see section 5) quantity $2a\beta n\Delta E$ represents a dimensionless thickness of the slab. Using the language of the *linear transport theory* one might say that uniqueness holds when the thickness of the slab is smaller than certain critical value. In addition, for future references, the existence part of the proof of Theorem 4.1 will be given for the system (4.1-4.5) with $\mu\frac{\partial I}{\partial x}$ replaced by its $(N+1)$-dimensional extension, namely $\Omega\cdot\nabla_x I$, and the corresponding boundary conditions. (In fact, existence proofs in both cases are the same.)

Let $\mathcal{O}$ be a smooth bounded convex open subset of $\mathbb{R}^{N+1}$ and $\mathcal{S}^N$ the unit sphere of $\mathbb{R}^{N+1}$, $N \ge 0$, with elements of $\mathcal{S}^N$ denoted by Ω. For each $x \in \partial\mathcal{O}$, $\eta(x)$ will denote the unit outward normal to $\partial\mathcal{O}$ at x and

$$(\partial\mathcal{O}\times\mathcal{S}^N)_- = \{(x,\Omega) \in \partial\mathcal{O}\times\mathcal{S}^N : \Omega\cdot\eta(x) < 0\}.$$

Proposition 4.3. *(Proposition 1 of [13]) For* $\mathcal{O} \subset \mathbb{R}^{N+1}$, *an open, connected set, let* $u \in L^p(\mathcal{O}\times\mathcal{S}^N)$, $p > 1$, $\Omega\cdot\nabla_x f \in L^p(\mathcal{O}\times\mathcal{S}^N)$ *and* $u|_{(\partial\mathcal{O}\times\mathcal{S}^N)_-} = k \in L^\infty((\partial\mathcal{O}\times\mathcal{S}^N)_-)$. *Then* $\int_{\mathcal{S}^N} u(t,x,\Omega)\, d\Omega \in W^{m,p}(\mathcal{O})$ *for any* $0 < m < \inf(1/p, 1-1/p)$, *and*

$$\left|\left|\int_{\mathcal{S}^N} u(x,\Omega)\, d\Omega\right|\right|_{W^{m,p}} \le C(k)\left[||u||_{L^p} + ||\Omega\cdot\nabla_x u||_{L^p}\right]. \tag{4.6}$$

In Proposition 4.3, $W^{m,p}(\mathcal{O})$ denotes the Sobolev space of functions whose i-order distributional derivatives, $i = 0, 1, \ldots, m$, belong to $L^p(\mathcal{O})$. We also note that in one-dimensional case $\mathcal{O} = (-a,a)$ and $\mathcal{S}^N = [-1,1]$, corresponding to $N = 0$.

From our earlier analysis (see, equations (3.4–3.7)) and under the assumptions of Theorem 4.1, it is clear that system (4.1–4.5) has a solution if the following proposition holds:

Proposition 4.4. *Let $\mathcal{I} = \int_{\mathcal{S}^N} I(x,\Omega)\,\delta\Omega$ with $\delta\Omega = d\Omega/|\mathcal{S}^N|$, the normalized measure on $\mathcal{S}^N$, and $\widetilde{\Gamma}(\mathcal{I}) = \alpha\beta n\Delta E/(\alpha + 2\beta\mathcal{I})$. For $0 \le k \in L^\infty((\partial\mathcal{O}\times\mathcal{S}^N)_-)$ the equation*

$$\begin{aligned} \Omega\cdot\nabla_x I &= \widetilde{\Gamma}(\mathcal{I})(\mathcal{I} - I), \\ I|_{(\partial\mathcal{O}\times\mathcal{S}^N)_-} &= k \end{aligned} \tag{4.7}$$

has a solution $I \in L^\infty(\mathcal{O}\times\mathcal{S}^N)$ with $0 \le I(x,\Omega) \le \|k\|_{L^\infty}$ and $\Omega\cdot\nabla_x I \in L^\infty(\mathcal{O}\times\mathcal{S}^N)$.

We remark that the boundary condition in (4.7) is understood in the sense of trace class (see, for example, [11–12]). Also, as noted earlier, problem (4.7) is an $(N+1)$-dimensional version of equation (3.8).

Sketch of the proof of Proposition 4.4: First, as in [13], for any $\lambda > 0$ and $0 \le V \in L^\infty(\mathcal{O})$ we consider the linear problem in U:

$$\begin{aligned} \lambda U + \Omega\cdot\nabla_x U &= \widetilde{\Gamma}(V)(\mathcal{U} - U), \\ U|_{(\partial\mathcal{O}\times\mathcal{S}^N)_-} &= k, \end{aligned} \tag{4.8}$$

whit $\mathcal{U} = \int_{\mathcal{S}^N} U\,\delta\Omega$. Using the standard results from the linear transport theory that go back to E. Hopf and K. M. Case (see, for example, [12] for overview) (4.8) has a unique solution $U \in L^p(\mathcal{O}\times\mathcal{S}^N)$, $p \ge 1$, with $\Omega\cdot\nabla_x U \in L^p(\mathcal{O}\times\mathcal{S}^N)$. Another way to look at solvability of problem (4.8) is to notice that $\Omega\cdot\nabla_x U + \widetilde{\Gamma}(V)(U - \mathcal{U})$ is m-accretive in L^1 since it is a perturbation of m-accretive operator $\Omega\cdot\nabla_x U$ in L^1 (with the boundary condition as in (4.8)) by accretive and Lipschitz (in L^1) operator $\widetilde{\Gamma}(V)(U - \mathcal{U})$.

Next, it is easy to show that U has the property

$$0 \le U \le \|k\|_{L^\infty}, \ \text{ almost everywhere in } \ (x,\Omega) \in \mathcal{O}\times\mathcal{S}^N. \tag{4.9}$$

Indeed, if (U_1,k_1) and (U_2,k_2) are solutions of (4.8) with $0 \le k_1 \le k_2$, almost everywhere, then from

$$\int\limits_{\mathcal{O}\times\mathcal{S}^N} \Big\{\lambda(U_1 - U_2) + \Omega\cdot\nabla_x[(U_1 - U_2)) + \widetilde{\Gamma}(V)[(U_1 - U_2) - (\mathcal{U}_1 - \mathcal{U}_2)]\Big\}\operatorname{sign}^+(U_1 - U_2)\,dxd\delta = 0,$$

and accretivness of $\Omega\cdot\nabla_x U$ and $\widetilde{\Gamma}(V)(U - \mathcal{U})$ (meaning that the second and the third integrals, above, are nonnegative) we obtain that

$$\lambda \int\limits_{\mathcal{O}\times\mathcal{S}^N} (U_1 - U_2)^+\,dxd\delta \le 0.$$

Equivalently, $U_1 \le U_2$ almost everywhere. This shows that the solution of (4.8) is nonnegative (take $U_1 = 0$, $k_1 = 0$, and $k_2 = k$). Similarly, from

$$\begin{aligned} \lambda \int\limits_{\mathcal{O}\times\mathcal{S}^N} \Big\{&(U - \|k\|_{L^\infty}) + \Omega\cdot\nabla_x(U - \|k\|_{L^\infty}) \\ &+ \widetilde{\Gamma}(V)[(U - \|k\|_{L^\infty}) - (\mathcal{U} - \|k\|_{L^\infty})]\Big\}\operatorname{sign}^+(U - \|k\|_{L^\infty}))\,dxd\delta = -\lambda\|k\|_{L^\infty} \le 0 \end{aligned}$$

and the boundary condition we obtain as before

$$\lambda \int_{\mathcal{O}\times\mathcal{S}^N} (U - \|k\|_{L^\infty})^+ \, dx d\delta \le 0,$$

or equivalently property (4.9).

Next, on the convex set $D = \{V \in L^p(\mathcal{O}) : 0 \le V \le \|k\|_{L^\infty}\}$, $1 < p < \infty$, we define the operator

$$\mathcal{A}(V) = \int_{\mathcal{S}^N} U \, d\Omega = \mathcal{U}, \tag{4.10}$$

where U is the unique solution of (4.8). Because of (4.9), $\mathcal{A}$ maps D into D. Furthermore, for any $V \in D$, the solution U of (4.8) satisfies

$$|\Omega \cdot \nabla_x U| \le (\lambda + 2\beta n \Delta E) \, \|U\|_{L^\infty}, \tag{4.11}$$

almost everywhere in (x, Ω), and thus $\Omega \cdot \nabla_x U \in L^p(\mathcal{O}\times\mathcal{S}^N)$, uniformly on the set D. Thus, Proposition 4.3 yields (since the imbedding $W^{m,p}(\mathcal{O}\times\mathcal{S}^N) \subset L^p(\mathcal{O}\times\mathcal{S}^N)$ is compact) that the set $\mathcal{A}(D)$ is relatively compact in $L^p(\mathcal{O}\times\mathcal{S}^N)$. Finally, since the set D is closed in $L^p(\mathcal{O}\times\mathcal{S}^N)$ and the map $\mathcal{A} : D \to D$ is continuous (again one can use (4.11) and Proposition 4.3), Schauder's fixed point theorem shows that, for each $\lambda > 0$, $\mathcal{A} : D \to D$ has a fixed point $\mathcal{I}_\lambda \in D$, with I_λ a solution of

$$\begin{aligned} \lambda I_\lambda + \Omega \cdot \nabla_x I_\lambda &= \widetilde{\Gamma}(\mathcal{I}_\lambda)(\mathcal{I}_\lambda - I_\lambda), \\ I_\lambda|_{(\partial\mathcal{O}\times\mathcal{S}^N)_-} &= k. \end{aligned} \tag{4.12}$$

In order to complete the proof of Proposition 4.4 we take $\lambda \to 0$. Using (4.9) and (4.11), together with Proposition 4.3, after extracting a subsequence, we obtain, for $p \ge 1$,

$$\mathcal{I}_\lambda \longrightarrow \mathcal{I} \qquad \text{strongly in } L^p(\mathcal{O}) \text{ as } \lambda \to 0, \tag{4.13}$$

$$I_\lambda \longrightarrow I \qquad \text{weakly in } L^p(\mathcal{O}\times\mathcal{S}^N) \text{ as } \lambda \to 0, \tag{4.14}$$

$$\Omega \cdot \nabla_x I_\lambda \longrightarrow \Omega \cdot \nabla_x I \qquad \text{weakly in } L^p(\mathcal{O}\times\mathcal{S}^N) \text{ as } \lambda \to 0, \tag{4.15}$$

Next, since the function $\widetilde{\Gamma}$ is bounded, we also have

$$\widetilde{\Gamma}(\mathcal{I}_\lambda)\mathcal{I}_\lambda \longrightarrow \widetilde{\Gamma}(\mathcal{I})\mathcal{I} \qquad \text{strongly in } L^p(\mathcal{O}) \text{ as } \lambda \to 0, \tag{4.16}$$

$$\widetilde{\Gamma}(\mathcal{I}_\lambda)I_\lambda \longrightarrow \widetilde{\Gamma}(\mathcal{I})I \qquad \text{weakly in } L^p(\mathcal{O}\times\mathcal{S}^N) \text{ as } \lambda \to 0, \tag{4.17}$$

Finally, using the limits (4.13–4.17), the passage to the limit in equation (4.12) can be accomplished, thus showing that I is a solution of problem (4.7). In addition, since $0 \le I_\lambda \le \|k\|_{L^\infty}$ for each $\lambda > 0$, the weak limit of I_λ also satisfies this inequality.

In order to complete the proof of Theorem 4.1 we need to show that the functions n_1, n_2, T, and $\mathcal{I}$ are Hölder continuous on $[-a, a]$. In view of (3.5)–(3.7), it is enough to prove this property for the integrated intensity $\mathcal{I}$. Using both (4.9) and (4.11), applied to I, we have, for $r = \sqrt{|x-y|} > 0$,

$$\left| \int_{-1}^{1} I(x,\mu)\,d\mu - \int_{-1}^{1} I(y,\mu)\,d\mu \right| \leq \int\limits_{|\mu|\leq r} |I(x,\mu) - I(y,\mu)|\,d\mu + \int\limits_{|\mu|\geq r} \frac{1}{|\mu|} \int_{x}^{y} \left| \mu \frac{\partial I}{\partial x} \right| d\mu$$

$$\leq \max\{2, 4\beta n \Delta E\} \|k\|_{L^\infty} \left[r + \frac{|x-y|}{r} \right] \leq 4 \max\{1, 2\beta n \Delta E\} \|k\|_{L^\infty} |x-y|^{\frac{1}{2}}, \tag{4.18}$$

This completes the proof of Theorem 4.1.

Sketch of the proof of Theorem 4.2: In view of representation (3.5–3.7) for n_1, n_2, and T in terms of $\mathcal{I}$, it is enough to prove the uniqueness of I, a solution of (3.8), or equivalently a solution of equation (4.7). First, we notice that if I_1 and I_2 are solutions (4.7) then

$$\begin{aligned} \mu \frac{\partial (I_1 - I_2)}{\partial x} &= \widetilde{\Gamma}(\mathcal{I}_1) \left[\frac{(\alpha + 2\beta I_2)}{(\alpha + 2\beta \mathcal{I}_2)} (\mathcal{I}_1 - \mathcal{I}_2) - (I_1 - I_2) \right], \\ (I_1 - I_2)(-a,\mu) &= 0, \quad \forall \mu > 0, \\ (I_1 - I_2)(a,\mu) &= 0. \quad \forall \mu < 0, \end{aligned} \tag{4.19}$$

where $4\pi \mathcal{I}(x) = \int_{-1}^{1} I(x,\mu)\,d\mu$. Before proceeding further, we want to point out that uniqueness in our case can be proven if the following linear transport theory problem

$$\begin{aligned} \mu \frac{\partial \Phi}{\partial x} + (\beta n \Delta E)\, \sigma_0(x) \left[\Phi - \frac{\sigma_1(x,\mu)}{4\pi} \int_{-1}^{1} \Phi(x,\mu)\,d\mu \right] &= 0, \\ \Phi(-a,\mu) = 0, \quad \forall \mu > 0&, \\ \Phi(a,\mu) = 0. \quad \forall \mu < 0&, \end{aligned} \tag{4.20}$$

has only the trivial solution (among bounded functions). In (4.20), $\sigma_0(x) = \alpha/(\alpha + 2\beta \mathcal{I}_1)$ and $\sigma_1(x,\mu) = (\alpha + 2\beta I_2)/(\alpha + 2\beta \mathcal{I}_2)$. The problem of finding a nontrivial solution of (4.20) is the well known *critical condition* problem in the neutron transport theory. In fact, (4.20) can be viewed as an eigenvalue problem for $\beta n \Delta E$ in terms of the *critical* thickness of the slab.

Now, going back to our original proof, we see that after integrating (4.19) with respect to x and using the boundary conditions we obtain (as in (3.9))

$$\begin{aligned} &I_1(x,\mu) - I_2(x,\mu) = \\ &\Theta(\mu) \int_{-a}^{x} \widetilde{\Gamma}(\mathcal{I}_1(x')) \left(\frac{\alpha + 2\beta I_2(x',\mu)}{\alpha + 2\beta \mathcal{I}_2(x')} \right) [\mathcal{I}_1(x') - \mathcal{I}_2(x')] \frac{1}{\mu} \exp\left[-\int_{x'}^{x} \frac{1}{\mu} \widetilde{\Gamma}(\mathcal{I}_1(x''))dx'' \right] dx' \\ &-\Theta(-\mu) \int_{x}^{a} \widetilde{\Gamma}(\mathcal{I}_1(x')) \left(\frac{\alpha + 2\beta I_2(x',\mu)}{\alpha + 2\beta \mathcal{I}_2(x')} \right) [\mathcal{I}_1(x') - \mathcal{I}_2(x')] \frac{1}{\mu} \exp\left[\int_{x}^{x'} \frac{1}{\mu} \widetilde{\Gamma}(\mathcal{I}_1(x''))dx'' \right] dx', \end{aligned} \tag{4.21}$$

where Θ is the Heaviside function. After further integration of (4.21) with respect to (x,μ) and noticing that $\widetilde{\Gamma}(\mathcal{I}_1) \geq \alpha\beta n \Delta E/(\alpha + 2\beta \|k\|_{L^\infty}) = \gamma$ we have the inequality

$$\|I_1 - I_2\|_{L^1} \leq \frac{\beta n \Delta E}{2\pi} \left(1 + \frac{2\beta}{\alpha} \|k\|_{L^\infty} \right) \|I_1 - I_2\|_{L^1} \sup_{|x'|\leq a} \int_{-a}^{a} E_1(\gamma|x-x'|)\,dx, \tag{4.22}$$

where $E_1(\xi)$, given in (3.10b), is a well known function in the transport theory that has a logarithmic singularity at $\xi = 0$. Next, using the Schwartz inequality and tables of integrals we obtain

$$\begin{aligned}\sup_{|x'|\le a}\int_{-a}^{a} E_1(\gamma|x-x'|)\,dx &\le \sqrt{2a}\sup_{|x'|\le a}\left(\int_{-\infty}^{\infty} E_1^2(\gamma|x-x'|)\,dx\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2a}{\gamma}}\left(\int_{-\infty}^{\infty} E_1^2(|x|)\,dx\right)^{\frac{1}{2}} = 2\sqrt{\frac{2a\log 2}{\gamma}}.\end{aligned} \tag{4.23}$$

Combination of (4.22) and (4.23) results in the inequality

$$\|I_1 - I_2\|_{L^1} \le \frac{\sqrt{2\log 2}}{\pi}\,(a\beta n\Delta E)^{\frac{1}{2}}\left(1+\frac{2\beta}{\alpha}\|k\|_{L^\infty}\right)^{\frac{3}{2}}\|I_1 - I_2\|_{L^1}, \tag{4.24}$$

which shows that $\|I_1 - I_2\|_{L^1} = 0$, or equivalently $I_1 = I_2$ in L^1, if $a\beta n\Delta E$ is sufficiently small. This completes the proof of Theorem 4.2.

Finally, we note that the same estimations (4.22–4.23) can be applied to the linear *critical condition* problem (4.20), thus showing that it has only trivial solutions (among bounded functions) when $a\beta n\Delta E$ is sufficiently small.

5. NUMERICAL SIMULATIONS

We start by rewriting equations (2.2-2.4) in a dimensionless and rescaled form. Next, by introducing

$$\widetilde{T} = k_B T/(h\nu)\ , \quad \widetilde{n}_i = n_i/n\ , \quad \widetilde{I} = I/\widehat{I}\ ,$$

where $\widehat{I}$ is a reference intensity, we can rewrite the model equations in the form

$$\frac{\partial \widetilde{T}}{\partial \widetilde{t}} = \frac{2}{3}[\widetilde{n}_1\theta_1 + \widetilde{n}_2\theta_2][\widetilde{n}_2 - \widetilde{n}_1\exp(-1/\widetilde{T})]\ , \tag{5.1}$$

$$\frac{\partial \widetilde{n}_2}{\partial \widetilde{t}} = [\widetilde{n}_1\theta_1 + \widetilde{n}_2\theta_2][\widetilde{n}_1\exp(-1/\widetilde{T}) - \widetilde{n}_2] + \eta[\widetilde{\mathcal{I}}(\widetilde{n}_1 - \widetilde{n}_2) - 4\pi\lambda\widetilde{n}_2]\ , \tag{5.2}$$

$$\frac{\partial \widetilde{I}}{\partial \widetilde{t}} + \mu\frac{\partial \widetilde{I}}{\partial \widetilde{x}} = \lambda\widetilde{n}_2 - \widetilde{I}(\widetilde{n}_1 - \widetilde{n}_2)\ , \tag{5.3}$$

where

$$\theta_i = \gamma_{i2}/(\beta ch\nu)\ , \quad \eta = \widehat{I}/(ch\nu n)\ , \quad \lambda = \alpha/(\beta\widehat{I})\ ,$$

and

$$\widetilde{\mathcal{I}} = \mathcal{I}/\widehat{I}\ , \quad \widetilde{t} = \beta ch\nu nt\ , \quad \widetilde{x} = \beta h\nu nx\ .$$

By choosing $\widehat{I} = \alpha/\beta$, we also have $\lambda = 1$ and $\eta = \alpha/(\beta ch\nu n)$. We note that there are other choices for $\widehat{I}$ that could be more suitable in different contexts. Hereinafter, for simplicity, we shall drop the tilde from the dimensionless variables and, in addition, we assume that γ_{12} and γ_{22} are constants. This corresponds to the case of Maxwellian inelastic interactions.

We have performed numerical simulations in order to compare the present model with the Milne–Chandrasekhar problem. In particular, the goals of the simulations were:

a) comparisons between the present model and the one from Remark 1 (Chandrasekhar's model, without inelastic interactions),

b) comparisons of the asymptotic solutions obtained from numerical simulations with the approximate (Eddington) analytical solution (3.16).

The numerical simulations have been performed by using the "splitting" method described in [10]. The method consists in splitting each time–step of integration into two substeps. In the first one, we integrate equation (5.3) in each space–node with an upwind finite difference scheme [10], by setting its right–hand–side equal to zero (free streaming). In the second substep equations (5.1, 5.2) and equation (5.3) (without the convection term) are then integrated by means of a standard 4th–order Runge–Kutta routine for each space–node.

The numerical results are presented in figures 1 through 3. First of all, we observe that the final states for $\mathcal{I}$ and n_2 (please note Chandrasekhar's model does not include variations of T) are the same for the two models we consider, since they depend only on the gas–radiation interaction terms. In Chandrasekhar's model, driven only by means of gas–radiation interaction, the final state is reached faster than in our model, that includes additional driving mechanism of inelastic interaction.

Numerical results are given for two meaningful cases at different initial temperatures (in both cases the impinging radiation I^* is constant with respect to μ):

- Case 1 (Fig.1): low temperature, which implies $I(0,x) < I^*$
- Case 2 (Fig.2): high temperature, which implies $I(0,x) > I^*$.

Note that by putting together inequalities (2.1), including the rescaling done at the beginning of this section, one obtains

$$\frac{h\nu\pi}{8mc^2} \;<<\; T \;<<\; \frac{mc^2}{h\nu} \;.$$

Therefore considering a mercury gas illuminated by a mercury arc (see, for example, [5] and [8]) such inequalities are well satisfied by rescaled temperature values of the order of unity. Results in figures 1 and 2 are shown at three different times: beginning of the transient state (a), during the transient state (b), at quasi–asymptotic state (c). In particular, we plot $J = \mathcal{I}$, $n2 = n_2$ and T versus $x \in [-a, a]$. At each time instant we give for reference the maximum and minimum values for J, $n2$ and T, as computed by means of our model.

Finally, in figure 3 we show the comparison between the asymptotic state of J, calculated numerically, and its approximate analytical solution, obtained by Eddington expansion. As expected we find a good agreement between the two solutions within the slab. The only observable differences can be found at both the boundaries (especially the right one), where the intensity is more strongly anisotropic.

Acknowledgments : The present piece of research has been partially supported by the Italian Ministry of University (MURST) and by the National Group of Mathematical Physics of the National Research Council (CNR–GNFM).

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