Summer Institute I – 2006

Jacek Polewczak

Some interesting and unusual mathematical modeling problems and their solutions

Problem 1.

The graph of a certain function $y = f(x)$ passes through the origin. If (x, y) is a point ($\neq (0,0)$) on the graph in the first quadrant, then the graph divides the rectangle with vertices $(0,0)$, $(x,0)$, (x,y), and $(0,y)$ into two parts A and B so that the area of A is n times the area of B ($n > 1$, integer). Find $f(x)$.

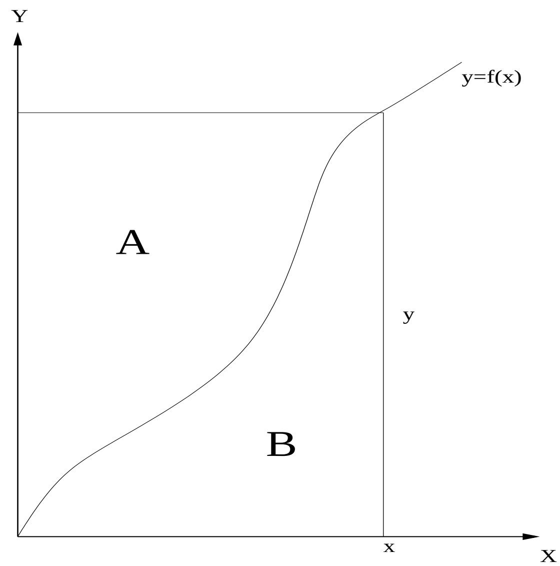

Figure 1. Illustration to Problem 1.

From Figure 1 we have that area of $A \cup B$ is given by

$$(\text{area of } A \cup B) = xy = xf(x) = (\text{area of } A) + (\text{area of } B) = n\int_0^x f(t)\,dt + \int_0^x f(t)\,dt = (n+1)\int_0^x f(t)\,dt.$$

After differentiating both sides of the identity

$$xf(x) = (n+1)\int_0^x f(t)\,dt,$$

we see that $f(x)$ satisfies the differential equation

$$f(x) + xf'(x) = (n+1)f(x), \qquad \Longrightarrow \qquad xf'(x) - nf(x) = 0$$

The last equation is the first order linear differential equation that can be solved explicitly. Its solution is $f(x) = cx^n$.

Remark 1. *If one interchanges the roles of A and B in Figure 1, i.e., set the area of $A \cup B$ equal to*

$$xf(x) = \int_0^x f(t)\,dt + \frac{1}{n}\int_0^x f(t)\,dt,$$

then one gets the answer $f(x) = cx^{1/n}$.

Problem 2.

The graph of a certain function $y = f(x)$ passes through $(3, 2)$. Let $L(x, y)$ be the segment of the tangent line to the graph at (x, y) in the first quadrant. Suppose that each point (x, y) on the graph is the midpoint of $L(x, y)$. Find $f(x)$.

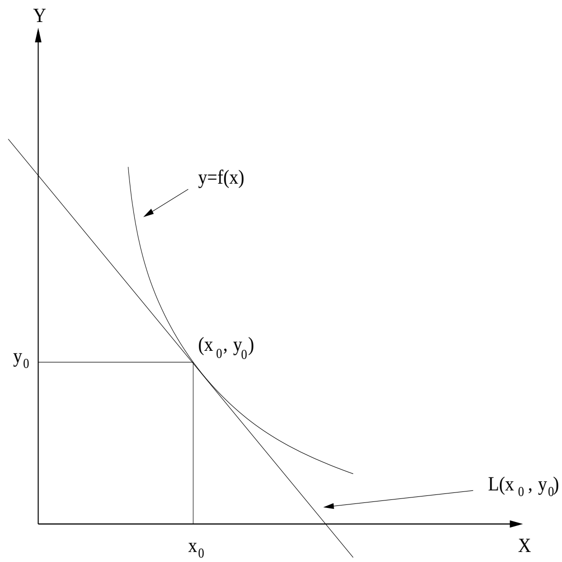

FIGURE 2. Illustration to Problem 2.

From Figure 2 we see that the tangent line has the x-intercept equal to $2x_0$ and the y-intercept equal to $2y_0$ (since (x_0, y_0) is the midpoint of the line segment $L(x_0, y_0)$). Furthermore, the slope of the tangent line is equal to $-\dfrac{2y_0}{2x_0} = f'(x_0)$. This true at any point (x, y) on the graph of $y = f(x)$. This also means that $f(x)$ satisfies the equation

$$y' = -\frac{y}{x}, \quad \text{or equivalently} \quad xy' + y = 0. \tag{1}$$

The solution of (1) is $y = c/x$. Now, if $(3, 2)$ is on the graph, then $c = 6$; thus $f(x) = 6/x$.

Problem 3.

A large snowball is shaped into the form of a sphere. Starting at some time, which we can designate as $t = 0$, the snowball begins to melt. We assume for the sake of discussion that the snowball melts in such a manner that its shape remains spherical. Discuss the quantities that change with time as the snowball melts. If possible, construct a mathematical model that describes the state of the snowball at any time.

The radius, volume, and surface are of the snowball change in time. As the snowball melts the volume decreases. In other words, the process of *melting* can be interpreted as the rate of change of the volume of the snowball in time.. Now, it seems *reasonable* to assume (**WHY?**) that the rate at which the snowball melts is *proportional* to the surface area. That is, $dV/dt = -kS$, where $k > 0$ is a proportionality constant. Now, $V = 4\pi r^3/3$ and $S = 4\pi r^2$, so

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = S\frac{dr}{dt} = -kS.$$

Thus, $dr/dt = -k$ is the differential equation describing the above process of snowball melting.

Problem 4.

Ralph Palmer Agnew's ***snowplow problem***

(from the text Differential Equations by Ralph Palmer Agnew, McGraw-Hill Book Co.)

One day it started snowing at heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?

We assume that the plow clears snow at a constant rate of k cubic miles per hour. Let t be the time in hours after noon, $x(t)$ the depths in miles of the snow at time t, and $y(t)$ the distance the plow has moved in t hours. The dy/dt is the velocity of the plow and the assumption (that plow clears snow at a constant rate) gives

$$wx\frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour.

Now let t_0 be the number of hours before noon when it started snowing and let $s > 0$ be the constant rate in miles per hour at which x increases. It means that $dx/dt = s$ and therefore, for $t > -t_0$, $x = s(t + t_0)$. (Note, that in our notation, $t = 0$ is the time the snoplow started working and $t = -t_0$ is the time it started snowing.) The differential equation becomes

$$\frac{dy}{dt} = \frac{k}{ws}\frac{1}{t+t_0}.$$

Integrating we obtain

$$y = \frac{k}{ws}\left[\ln(t+t_0) + c\right],$$

where c is a constant of integration. Now, when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$ (**Do you know why?**), we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. The solutions are $t_0 = \frac{-1\pm\sqrt{5}}{2}$. Since $t_0 > 0$, we have $t_0 = \frac{-1+\sqrt{5}}{2} \approx 0.618$ hours ≈ 37 minutes. Thus it started snowing at about $11:23$ in the morning.

Nice problem, isn't it !!!

Problem 5.

Suppose a hole is drilled through the center of the earth and a body of mass m is dropped into the hole. Discuss the possible motion of the mass. Construct a mathematical model that discribes the motion.

Notation:

Let the distance from the center of the earth to the mass at any time t be denoted by r, let M denote the mass of earth, let M_r denote the mass of that portion of the earth witin the sphere of radius r, and let δ denote the constant density of the earth.

A possible motion for the mass is that it oscillates back and forth around the center of the earth. The gravitational force on mass m is $F = -kM_rm/r^2$, where $r > 0$ and k is the gravitational constant. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$, we have $M_r = r^3M/R^3$ and

$$F = -k\frac{M_rm}{r^2} = -k\frac{mM}{R^3}r.$$

Now from Newton's second law, $F = ma = d^2r/dt^2$, thus we have

$$m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \qquad \text{or} \qquad \frac{d^2r}{dt^2} = -k\frac{M}{R^3}r.$$

Remark 2. *We will see later on that the solution to this differential equations is a periodic function of t.*