

EXAMPLES OF INCOMPLETE METRIC SPACES
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JACEK POLEWCZAK

EXAMPLE 1

Let \mathbb{Q} be the set of all rational numbers with the metric given by

$$d(x, y) = |x - y|,$$

for $x, y \in \mathbb{Q}$. Consider the sequence $\{x_n\}$ of rational numbers such that $x_1 = 1$ and $x_{n+1} = 2\frac{1+x_n}{2+x_n}$, for $n \geq 2$. The sequence $\{x_n\}$ is an increasing sequence of rational numbers such that $x_n = 2\frac{1+x_n}{2+x_n} \leq 2$, for $n \geq 1$. (*Can you check it?*) Next, one shows (*Can you do it?*) that $\lim_{n \rightarrow \infty} x_n = \sqrt{2} \notin \mathbb{Q}$ and thus $\{x_n\}$ is a Cauchy sequence in \mathbb{Q} that is not convergent in \mathbb{Q} . Thus (\mathbb{Q}, d) is not a complete metric space.

EXAMPLE 2

Let X be the set of all continuous real-valued functions on $[0, 1]$ and define a metric on X by

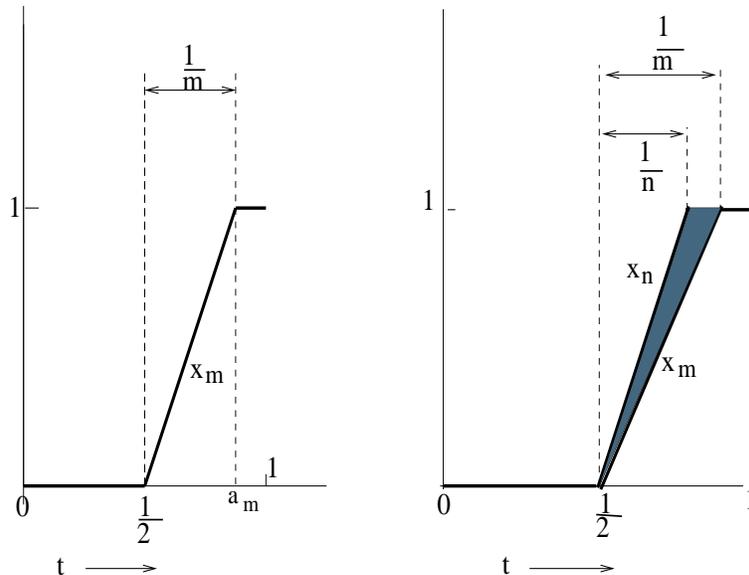
$$d(x, y) = \int_0^1 |x(t) - y(t)| dt,$$

for $x, y \in X$. The metric space (X, d) is not complete.

Proof. Define the sequence $x_m(t)$ of continuous functions on $[0, 1]$ by

$$x_m(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{2}; \\ m(x - \frac{1}{2}), & \text{if } \frac{1}{2} < t < a_m = \frac{1}{2} + \frac{1}{m}; \\ 1, & \text{if } a_m \leq t \leq 1. \end{cases}$$

$\{x_m\}$ is a Cauchy sequence. Indeed, $d(x_m, x_n)$ is the area of the shaded triangle in the figure below,



and

$$d(x_m, x_n) < \epsilon, \text{ when } n, m > 1/\epsilon.$$

Next, we show that x_m does not converge in X . For every $x \in X$ we have

$$d(x_m, x) = \int_0^1 |x_m(t) - x(t)| dt = \int_0^{1/2} |x(t)| dt + \int_{1/2}^{a_m} |x_m(t) - x(t)| dt + \int_{a_m}^1 |1 - x(t)| dt.$$

The integrands above are nonnegative, so is each integral on the right hand side. Therefore $d(x_m, x) \rightarrow 0$ as $m \rightarrow \infty$ would imply that each integral approaches zero, and, since $x(t)$ is continuous, we should have

$$x(t) = 0 \text{ if } t \in [0, 1/2) \quad \text{and} \quad x(t) = 1 \text{ if } t \in (1/2, 1].$$

But this is impossible for a continuous function. Hence $\{x_m\}$ does not converge. This proves that (X, d) is not a complete metric space. \square