We consider a tennis ball with mass $m$ and diameter $d$ moving in air near the earth surface. The moving ball is also spinning with angular velocity $\omega$. Vector $\omega$ has the direction of the axis of rotation and magnitude $\omega = |\omega| = d\phi(t)/dt = \dot{\phi}$, where $\phi(t)$ is an angle of rotation.

We use Cartesian coordinates system on the surface of the earth with $z$ axis directed vertically upwards. Next, we model the ball as a mass point moving under influence of three forces:

- The force of gravity $G = mg$, where $g = (0, 0, -g)$;
- The drag force $D = -D_L(v)(v/v)$, which has opposite direction to the velocity $v$;
- The Magnus force $M = M_L(v)[(\omega/\omega) \times (v/v)]$, where $\omega = |\omega|$ and $v = |v|$. Here, symbol $\times$ denotes the vector (cross) product of two vectors. The Magnus force is orthogonal to $\omega$ and $v$.

The magnitude of the drag force $D_L(v)$ and the Magnus force $M_L(v)$ are taken from the theory of ideal fluids:

$$D_L(v) = C_D \frac{1}{2} \frac{\pi d^2}{4} \rho v^2,$$

$$M_L(v) = C_M \frac{1}{2} \frac{\pi d^2}{4} \rho v^2,$$

where $\rho$ is air density. The coefficients $C_D$ and $C_M$ depend on the velocity $v$, the ball revolution and the material of the surface of the ball. These coefficients are determined from laboratory experiments. For a tennis ball in regions of velocities $v \in [13.6, 28] \text{[ms}^{-1}]$ and a ball revolution $n \in [800, 3250] \text{ rpm}$, the coefficients depend only on $v/w$, where the quantity $w = (d/2)|\omega \times (v/v)|$, which can be thought as the projection of the equilateral velocity $\omega(d/2)$ of the spinning ball onto the velocity vector $v$. Moreover, for a tennis ball we neglect the deceleration of the ball revolutions, so $w$ is assumed constant here. The results from laboratory experiments are:

$$C_D = 0.508 + \left( \frac{1}{22.053 + 4.196 \left( \frac{v}{w} \right)^{5/2}} \right)^{2/5},$$

$$C_M = \frac{1}{2.022 + 0.981 \left( \frac{v}{w} \right)}.$$

The trajectory of the tennis ball is governed by Newton’s second law with $\mathbf{r}(t) = (x(t), y(t), z(t))$:

$$m \frac{d^2 \mathbf{r}(t)}{dt^2} = -mg - D_L(v)(v/v) + M_L(v)[(\omega/\omega) \times (v/v)],$$

(1)

and the initial conditions

$$\mathbf{r}(0) = \mathbf{r}_0 \quad \text{and} \quad \frac{d\mathbf{r}(0)}{dt} = \mathbf{v}_0.$$

We will consider the most common case of topspin lob, for which vector of angular velocity $\omega$ lies in a horizontal plane and is orthogonal to $\mathbf{v}_0$ and (as it follows from (1)) to $\mathbf{v}(t)$, for $t \geq 0$.

This also implies that the trajectory of the tennis ball lies in the vertical plane. By choosing the $x$-axis in this plane, (1) have the form:

$$\ddot{x} = -C_D\alpha v \dot{x} + \eta C_M \alpha v \dot{z},$$

$$\ddot{z} = -g - C_D\alpha v \dot{z} - \eta C_M \alpha v \dot{x},$$

(2)
where \( v = \sqrt{\dot{x}^2 + \dot{z}^2} \) and \( \alpha = (\rho \pi d^2)/(8m) \). The parameter \( \eta = 1 \) for topspin lob and \( \eta = 0 \) for drag force only. The corresponding initial conditions at \( t = 0 \) are

\[
x(0) = 0, \quad z(0) = h, \quad \dot{x}(0) = v_0 \cos \theta, \quad \dot{z}(0) = v_0 \sin \theta,
\]

where \( v_0 \) is the magnitude of the initial vector \( \mathbf{v}_0 \) and \( \theta \) is an angle between \( \mathbf{v}_0 \) and \( x \)-axis.

Equations (2) form a nonlinear system of two differential equations and no analytical solution is known. The system can be solved numerically. For numerical methods is advantageous to transform (2) into a system of first order differential equations:

\[
\begin{align*}
\dot{x} &= v_x, \\
\dot{v}_x &= -C_D \alpha \mathbf{v} \cdot v_x + \eta C_M \alpha \mathbf{v} \cdot v_z, \\
\dot{z} &= v_z, \\
\dot{v}_z &= -g - C_D \alpha \mathbf{v} \cdot v_z - \eta C_M \alpha \mathbf{v} \cdot v_x,
\end{align*}
\]

(3)

where \( v = \sqrt{\dot{x}^2 + \dot{z}^2} \) and the initial condition become:

\[
x(0) = 0, \quad z(0) = h, \quad v_x(0) = v_0 \cos \theta, \quad v_z(0) = v_0 \sin \theta.
\]